UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 7

Due: March 3, 2017

1 Consider a morphism of schemes $f : X \to Y$. Prove that the following conditions are equivalent:

(a) f is locally of finite type.

(b) For every affine open $U \subset Y$, $f^{-1}(U)$ has an open covering by affine opens $(V_i)_i$ such that $\mathcal{O}_X(V_i)$ is a finitely generated $\mathcal{O}_Y(U)$-algebra for all i.

(c) For every affine open $U \subset Y$ and every affine open $V \subset f^{-1}(U)$, $\mathcal{O}_X(V)$ is a finitely generated $\mathcal{O}_Y(U)$-algebra.

2 Let $f : X \to Y$ be an immersion. Show that f can be factored as $f = j \circ i$, where j is an open immersion, and i is a closed immersion.

Remark. It is not true in general that f also factors as $f = i \circ j$ where i is a closed immersion, and j an open immersion. (Although this is true under mild assumptions.) Here is an example which you might want to think about (no need to hand it in).

Let k be a field and $X = \mathbb{A}_k^\infty = \mathrm{Spec}(k[x_1, x_2, \ldots])$. Let $j : U = \cup_{n\in\mathbb{N}} D(x_n) \to X$ be the open immersion. On $D(x_n)$ consider the closed immersion $Z_n \to D(x_n)$ defined by the ideal

$$\langle x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots\rangle \subset k[x_1, x_2, \ldots][1/x_n].$$

It is easy to see that these glue to a closed immersion $i : Z \to U$. Hence we get an immersion $f = j \circ i$. Why is there no factorization of f as an open immersion followed by a closed immersion? (*Hint: use the description of closed subschemes of an affine scheme given in class.*)

3 Let R be a ring and $p \in \mathbb{N}$ a prime number. R is said to be *of characteristic* p if $p \cdot 1_R = 0$ in R. A scheme X is said to be *of characteristic* p if for every open $U \subset X$, $\mathcal{O}_X(U)$ is of characteristic p.

(a) Prove that the following conditions are equivalent:

i. X is of characteristic p.

ii. For every $x \in X$, $\mathcal{O}_{X,x}$ is of characteristic p.

iii. $\mathcal{O}_X(X)$ is of characteristic p.

iv. There exists a morphism of schemes $X \to \mathrm{Spec}(\mathbb{F}_p)$, where the latter denotes the finite field with p elements.

(b) Let X be a scheme of characteristic p. Prove that there is a unique morphism of schemes $F_p : X \to X$ such that

- F_p is the identity on the underlying topological space.
- For any open $U \subset X$, F_p acts on $\mathcal{O}_X(U)$ by $r \mapsto r^p$.

F_p is called the *absolute Frobenius*.

(c) Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. For a finite type $\mathbb{F}_p$-scheme X, define a canonical morphism

$$X(\mathbb{F}_{p^e}) \to X(\overline{\mathbb{F}}_p)^{F_p^e}$$

where the right hand side denotes the subset of morphisms $f : \mathrm{Spec}(\overline{\mathbb{F}}_p) \to X$ such that $f = F_p \circ \cdots \circ F_p \circ f$ (F_p is composed e times). Prove that the map is bijective.