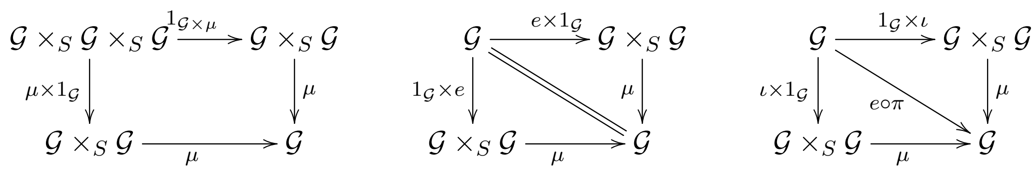
UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 6

Due: February 24, 2017

1 Consider the morphism $\mathrm{Spec}(\mathbb{Z}[\sqrt{-3}]) \to \mathrm{Spec}(\mathbb{Z})$ induced by the canonical inclusion of rings. Compute the (scheme-theoretic) fibers of this morphism and draw a picture of $\mathrm{Spec}(\mathbb{Z}[\sqrt{-3}])$ lying above $\mathrm{Spec}(\mathbb{Z})$. (We can think of this as a "two-sheeted cover" because of the similarities with, say, $\mathbb{C}[x] \xrightarrow{x\mapsto x^2} \mathbb{C}[x]$.)

2 Let S be a non-empty scheme. Define a *group scheme over* S as a "group in the category of S-schemes". In other words, it is an S-scheme $\pi : \mathcal{G} \to S$ together with an S-morphism $\mu : \mathcal{G} \times_S \mathcal{G} \to \mathcal{G}$ such that there exist S-morphisms $e : S \to \mathcal{G}$ and $\iota : \mathcal{G} \to \mathcal{G}$ such that the following diagrams commute:



A *morphism of group schemes over* S is a morphism of S-schemes compatible with the multiplication morphism.

(a) Define the group scheme GL_n for $n \geq 1$ (here $S = \mathrm{Spec}(\mathbb{Z})$) and the determinant morphism $\det : \mathrm{GL}_n \to \mathbb{G}_m := \mathrm{GL}_1$.

(b) The *kernel* of a morphism of group schemes $\varphi : \mathcal{G} \to \mathcal{H}$ over S is the fiber product $\mathcal{G} \times_\mathcal{H} S$ of φ and e. (It follows formally that this is again a group scheme over S.) Define $\mathrm{SL}_n = \ker(\det : \mathrm{GL}_n \to \mathbb{G}_m)$. Verify that the group $\mathrm{SL}_n(\mathbb{R}) = \hom_{\mathcal{SCH}}(\mathrm{Spec}(\mathbb{R}), \mathrm{SL}_n)$ is what you would expect. What is the dimension of SL_n (as a scheme)?

(c) Let G be a finite group, and $S = \mathrm{Spec}(k)$ the spectrum of a field. Associate to G a group scheme $\mathcal{G}$ over k such that for any connected k-scheme X, $\hom_k(X, \mathcal{G}) = G$. What is the ring of regular functions on $\mathcal{G}$?

3 Let $A = \oplus_{d\geq 0} A_d$ be a graded ring.

(a) Show that there exists a canonical morphism of schemes

$$\pi : \mathrm{Spec}(A)\backslash\mathcal{Z}(A_{>0}) \to \mathrm{Proj}(A)$$

such that for any $f \in A_{>0}$ homogeneous, $\mathrm{Spec}(A[1/f])$ is the fiber product of π and $\mathrm{Spec}(A[1/f]_0) \cong D(f) \to \mathrm{Proj}(A)$.

(b) Prove that the map on topological spaces induced by π is a quotient map.

(c) Assume that the irrelevant ideal is generated in degree 1, and let $\mathfrak{p} \in \mathrm{Proj}(A)$. Show that the fiber of π over $\mathfrak{p}$ is isomorphic to $\mathrm{Spec}(\kappa(\mathfrak{p})[x, x^{-1}])$.

This completes the analogy with the picture for varieties where we defined $\mathbb{P}^n_k$ in terms of the canonical projection $\pi : \mathbb{A}^{n+1}_k\backslash\{0\} \to \mathbb{P}^n_k$ whose fibers are punctured lines.