UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 2

Solution to problem 4

4 (a) Let A and B be two affine k-algebras. Show that $A \otimes_k B$ is a coproduct of A and B in the category of affine k-algebras.

Proof. Denote by i_A (resp. i_B) the canonical k-algebra morphism $A \to A \otimes_k B$ (resp. $B \to A \otimes_k B$). The image of i_A and i_B together generate $A \otimes_k B$ as a k-algebra hence $A \otimes_k B$ is finitely generated. I will prove below that $A \otimes_k B$ is a domain hence $A \otimes_k B \in \mathrm{AffAlg}/k$.
Given an affine k-algebra C and morphisms of k-algebras $\alpha : A \to C$ and $\beta : B \to C$, define $\gamma : A \times B \to C$ by $\gamma(a, b) = \alpha(a)\beta(b)$. Since α and β are k-linear morphisms, γ is k-bilinear and therefore passes to a k-linear morphism $\gamma : A \otimes_k B \to C$. Since α and β are algebra morphisms, γ is an algebra morphism as well.
By construction, we have $\gamma \circ i_A = \alpha$ and $\gamma \circ i_B = \beta$, and γ is uniquely determined by these two conditions since the image of i_A and i_B generate $A \otimes_k B$ as a k-algebra. This says precisely that $A \otimes_k B$ is the coproduct of A and B. □

Claim. *Let A and B be affine k-algebras. Then $A \otimes_k B$ is an affine k-algebra as well.*

Proof. We already remarked that $C = A \otimes_k B$ is finitely generated. Now let $c = \Sigma a_i \otimes b_i$ and $c' = \Sigma a'_j \otimes b'_j$ be elements of C and assume $cc' = 0$. Rearranging terms we may assume that the b_i (resp. the b'_j) are k-linearly independent. Denote by I the ideal in A generated by the a_i, and I' the ideal in A generated by the a'_j. Let $\mathfrak{m} \subset A$ be a maximal ideal. Under the morphism $\overline{(-)} : A \otimes_k B \to A/\mathfrak{m} \otimes_k B \cong B$ (by the Nullstellensatz)[1] we have $\overline{c}\overline{c'} = 0$ in B hence $\overline{c} = 0$ or $\overline{c'} = 0$. So either $I \subset \mathfrak{m}$, or $I' \subset \mathfrak{m}$. Since A is affine we must therefore have $I \subset \cap_{\mathfrak{m} \subset A \text{ maximal}} \mathfrak{m} = 0$ or $I' \subset \cap_{\mathfrak{m} \subset A \text{ maximal}} \mathfrak{m} = 0$ and so $c = 0$ or $c' = 0$. □

(b) Deduce that if $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are affine varieties then $X \times Y \subset \mathbb{A}^{m+n}$ with the induced topology is a product of X and Y in the category of varieties.

Proof. Let $\mathcal{O}(X) = k[x_1, \ldots, x_m]/I$ and $\mathcal{O}(Y) = k[y_1, \ldots, y_n]/J$ be the corresponding coordinate rings. By part (a), $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is a coproduct of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ in AffAlg/k and hence a product in $(\mathrm{AffAlg}/k)^{\mathrm{op}}$. Under the adjunction

$$\mathcal{O} : \mathrm{Var}/k \rightleftarrows (\mathrm{AffAlg}/k)^{\mathrm{op}} : \mathrm{mSpec}$$

described in class, the right adjoint mSpec takes products to products hence we reduce to prove that the coordinate ring of $X \times Y$ is isomorphic to $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$.
A polynomial $f \in R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ vanishes on $X \times Y$ if and only if it vanishes on the intersection of $X \times \mathbb{A}^n$ and $\mathbb{A}^m \times Y$, i.e. if and only if $f \in \sqrt{R \cdot I + R \cdot J}$. Now, the canonical morphism

$$k[x_1, \ldots, x_m]/I \otimes_k k[y_1, \ldots, y_n]/J \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]/(R \cdot I + R \cdot J)$$

is clearly an isomorphism, and we know that the left hand side is a domain so $R \cdot I + R \cdot J = \sqrt{R \cdot I + R \cdot J}$ and we win. □

[1] Here we use that k is algebraically closed. In fact, the statement is not true without this assumption. Consider for example $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$!