UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 1

Due: January 20, 2017

1 Let $X \subset \mathbb{A}^4$ be the algebraic set defined by the polynomials $xy - zw$, $xz - y^2$, $y(w-1)$. Find its irreducible components and corresponding prime ideals.

2 Show that for a topological space X, the following are equivalent:

(a) X is irreducible.

(b) The intersection of any two non-empty open subsets of X is non-empty (and open).

(c) Every non-empty open subset of X is dense.

(d) Every non-empty open subset of X is irreducible.

(e) The image of every continuous map $X \to Y$ is irreducible.

3 Consider the set $T = \{(t, ut^2, u^2t, u^3) \mid u, t \in \mathbb{A}^1\} \subset \mathbb{A}^4$. Show that T is an algebraic set and find its ideal $\mathcal{I}(T)$. Prove that T is in fact an affine variety. *Hint: For the last statement, part (e) of the previous exercise may be helpful.*

4 Let X be the affine variety $\mathcal{Z}(x^2 + y^2 - 1) \subset \mathbb{A}^2$ and consider the rational function $f = \frac{y+1}{x}$ on X. Determine at which points f is regular. *Hint: Beware of* $\operatorname{char}(k)$.

5 For this problem assume that $\operatorname{char}(k) = p > 0$. Define the *Frobenius morphism* $F_p : \mathbb{A}^n \to \mathbb{A}^n$ by $(x_1, \ldots, x_n) \mapsto (x_1^p, \ldots, x_n^p)$. Establish the following properties of F_p:

(a) It is indeed a morphism of affine varieties.

(b) It is a homeomorphism of topological spaces.

(c) It is not an isomorphism of affine varieties.