Homework 1

Solution to problem 3

3 Consider the set $T = \{(t, ut^2, u^2t, u^3) \mid u, t \in \mathbb{A}^1\} \subset \mathbb{A}^4$. Show that T is an algebraic set and find its ideal $\mathcal{I}(T)$. Prove that T is in fact an affine variety.

Remark. Almost everyone identified polynomials and proved that their zero set is equal to T so I will not do this here. To show that T is an affine variety, it suffices to note that it is the image of the morphism $\mathbb{A}^2 \to \mathbb{A}^4$ essentially given in the statement of the problem.

The tricky part is to show that the polynomials identified actually generate $\mathcal{I}(T)$. More explicitly, let $I \subset k[x, y, z, w]$ be the ideal generated by $g_1 = x^3w - yz$, $g_2 = yw - z^2$, $g_3 = x^3z - y^2$. The arguments referred to above establish that $\mathcal{Z}(I) = T$ and that $\sqrt{I}$ is a prime ideal. What we would like to show is that in fact $I = \sqrt{I}$. Since it doesn't seem more difficult to show that I is prime, we will do that instead. Establishing that an ideal is radical or prime tends to be tricky (although computer algebra systems are good at it).

Claim. *I is prime.*

Proof. Let $R = k[x, y, z, w]/I$. I will prove below that x is not a zerodivisor in R hence $R \subset R_x$ and it suffices to show that R_x is a domain. But

$$\begin{aligned} R_x &\cong k[x, x^{-1}, y, z, w]/\langle w - x^{-3}yz, yw - z^2, z - x^{-3}y^2\rangle \\ &\cong k[x, x^{-1}, y]/\langle y(x^{-3}yx^{-3}y^2) - (x^{-3}y^2)^2\rangle \\ &= k[x, x^{-1}, y], \end{aligned}$$

which is indeed a domain. □

If α is a monomial in x, y, z, w and $f \in A = k[x, y, z, w]$, I will say that f contains α if one of the terms of f is divisible by α. I will use the following fact.

Fact. *We can divide with remainder by g_1, g_2, g_3, i.e. every $f \in A$ can be represented as $f = (\Sigma_i p_i g_i) + r$ where r does not contain any of x^3w, yw, x^3z. (I will refer to an r satisfying these conditions as being in normal form.) Moreover, r is uniquely determined, and if $r \neq 0$ then $f \notin I$.*

Proof. This follows from the theory of Gröbner bases.[1] More specifically, it is easy to check that the g_1, g_2, g_3 satisfy Buchberger's criterion so they form a Gröbner basis (with respect to the deglex order). □

In particular, every element of R is represented by a polynomial in normal form. Thus let f be a polynomial in normal form.

[1] For a two page summary see `https://math.berkeley.edu/~bernd/what-is.pdf`

Claim. *If $xf = 0$ in R then $f = 0$ in R. (Hence x is not a zero-divisor in R.)*

Proof. Collect the terms of f into groups: $f = f_z + f_w + f_{ww} + g$ where

- f_z is those terms divisible by x^2z, but not by w,
- f_w is those terms divisible by x^2w, but not by w^2,
- f_{ww} is those terms divisible by x^2w^2,
- g is the remaining terms.

Then $xf = xf_z + xf_w + xf_{ww} + xg$ and applying the division algorithm we obtain the polynomial in normal form

$$\frac{f_z}{x^2z}y^2 + \frac{f_w}{x^2w}yz + \frac{f_{ww}}{x^2w^2}z^3 + xg,$$

which by our assumption and the fact mentioned above must equal 0. Using repeatedly that f_z, f_w, f_{ww}, g are in normal form we deduce:

- the last summand is the only one containing x hence $g = 0$;
- from the remaining ones, the first summand is the only one containing y^2 hence $f_z = 0$;
- the second summand is then the only one containing y thus $f_w = 0$;
- and this implies $f_{ww} = 0$ as well.

We conclude that $f = 0$ in R. □