THE SPECTRUM OF ARTIN MOTIVES OVER FINITE FIELDS

Abstract. In this short announcement we describe the spectrum of Artin motives over a finite field, and thereby classify them up to the tensor triangulated structure. Proofs will appear as part of forthcoming work on the tensor-triangular geometry of Artin-Tate motives.

1. Introduction

Fix a field $\mathbb{F}$ and consider Voevodsky's category of (geometric) motives of algebraic $\mathbb{F}$-varieties $\mathrm{DM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$. This is a tensor triangulated category, and a highly ambitious goal consists in classifying its objects up to the tensor triangulated structure. In other words, we view two objects in $\mathrm{DM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$ as equivalent if they can be constructed out of each other using extensions, suspensions, direct summands, and tensor products with arbitrary objects. Equivalently, we view two objects as equivalent if they generate the same thick tensor ideal in $\mathrm{DM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$.

This goal is at the moment far beyond our reach. (We refer to [Gal20] for some indications as to how ambitious this goal is.) Instead, the authors have come to see the subcategory of Artin-Tate motives $\mathrm{DATM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z}) \subseteq \mathrm{DM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$ as a particularly good first step towards that goal, in that it exhibits many of the complexities present in $\mathrm{DM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$ while still being somewhat tractable. Recall that $\mathrm{DATM}^{\mathrm{gm}}(\mathbb{F};\mathbb{Z})$ is generated as a triangulated subcategory by motives of the form $\mathrm{M}(\mathrm{Spec}(\mathbb{F}'))(n)$, where $\mathbb{F}'/\mathbb{F}$ is a finite separable field extension, and (n) denotes the nth Tate twist ($n \in \mathbb{Z}$). In particular, this category sees the motivic cohomology of the field $\mathbb{F}$, and is intricately linked with the representation theory of the absolute Galois group of $\mathbb{F}$ [BG20].

Classifying thick tensor ideals in a tensor triangulated category $\mathcal{K}$ is at the heart of tensor-triangular geometry [Bal10b]. Recall that the latter associates to $\mathcal{K}$ a topological space $\mathrm{Spc}(\mathcal{K})$, called the spectrum, whose underlying set consists of the prime ideals in $\mathcal{K}$ and whose topology encodes the (radical)[1] thick tensor ideals. Indeed, the latter correspond bijectively to the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ (that is, arbitrary unions of closed subsets whose complements are quasi-compact).

We intend to perform a systematic study of the tensor-triangular geometry of Artin-Tate motives. In this short note we wish to record an application of this study, namely the computation of the spectrum of Artin motives over finite fields.

2. Result

Fix $\mathbb{F}_q$ a field of positive characteristic, and consider the category $\mathrm{DAM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z})$ of Artin motives over $\mathbb{F}_q$, that is, the tensor triangulated subcategory of $\mathrm{DM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z})$ generated by motives of finite separable field extensions of $\mathbb{F}_q$. The spectrum $\mathrm{Spc}(\mathrm{DAM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z}))$ is fibered over $\mathrm{Spec}(\mathbb{Z})$ by general principles [Bal10a]. The

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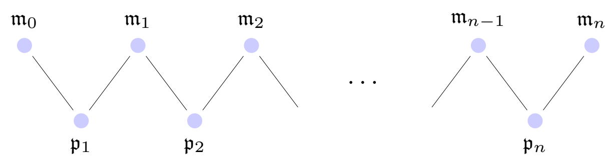
[1]In the examples we are interested in, every object admits a tensor dual, in which case thick tensor ideals are automatically radical.

fiber over (0) turns out to be a singleton space, and the fibers over the non-zero primes (ℓ) are all isomorphic. In the sequel we therefore fix one arbitrary such prime ℓ and we focus on $\mathrm{DAM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z}/\ell)$.

2.1. *Construction.* Let $\mathbb{W}_n$ denote the topological space of cardinality $2n+1$ with specialization relations pointing upwards:



There are obvious inclusions $\mathbb{W}_{n-1} \hookrightarrow \mathbb{W}_n$ (preserving the points' names) and we denote by $\mathbb{W}_\infty$ the colimit of these inclusions in the category of topological spaces:

$$\mathbb{W}_\infty := \operatorname*{colim}_n \mathbb{W}_n$$

2.2. **Theorem.** *Let ℓ be a non-zero prime. The spectrum of* $\mathrm{DAM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z}/\ell)$ *is the one-point compactification* $\hat{\mathbb{W}}_\infty$ *of* $\mathbb{W}_\infty$.

2.3. *Remark.* Very explicitly, the space $\hat{\mathbb{W}}_\infty$ consists of three types of points:

- points $\mathfrak{m}_n$ for $n \geq 0$;
- points $\mathfrak{p}_n$ for $n \geq 1$;
- a point ∞.

And the open subsets are of two types:

- subsets $U \subseteq \left\{ \mathfrak{m}_m, \mathfrak{p}_n \,\middle|\, m \geq 0, n \geq 1 \right\}$ such that if $\mathfrak{m}_0 \in U$ then $\mathfrak{p}_1 \in U$, and if $\mathfrak{m}_n \in U$ with $n \geq 1$ then $\mathfrak{p}_n, \mathfrak{p}_{n+1} \in U$;
- cofinite subsets $U \amalg \infty \subseteq \hat{\mathbb{W}}_\infty$ such that if $\mathfrak{m}_{n-1}, \mathfrak{m}_n \in U$ then $\mathfrak{p}_n \in U$.

2.4. **Corollary.** *The cardinality of the class of thick triangulated tensor-ideal subcategories of* $\mathrm{DAM}^{\mathrm{gm}}(\mathbb{F}_q;\mathbb{Z}/\ell)$ *is the continuum.*

References

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