Stable $\mathbb{A}^1$-homotopy theory

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Abstract

These are notes for a series of lectures delivered at the "Motives in Montpellier" winter school in January 2026. In them we provide a glimpse into Morel–Voevodsky's stable motivic homotopy theory.

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1 Introduction

This is a school on *motives* and you will be familiar with the idea that these should form the 'universal cohomology theory'. You will also be familiar with two common complaints from people outside (and sometimes inside) the field: that the theory of motives is purely conjectural, or quite to the contrary, that there are (too) many theories of motives. But surely the latter, and to some extent possibly the former too, simply stems from the ambiguity in the term 'cohomology theory': of what? valued in what? satisfying what?…

Algebraic topologists, on the other hand, find themselves in an enviable situation: there is an accepted notion of cohomology theory for spaces, these assemble into a category (the category of spectra), and for many decades mathematicians have been busy uncovering a beautiful and rich picture capturing what seems to be the essence of this object. It is therefore very natural to want to imitate this in algebraic geometry and to define a category of motivic spectra in parallel to its topological cousin. These notes are about such an attempt that started with the work of Morel–Voevodsky [MV99] and has been pursued by many others in the meantime.[1]

Even at the surface level, spectra have some particularly attractive features:

(i) as mentioned, they classify cohomology theories

(ii) they admit a rather explicit description connecting them to spaces

(iii) the category of spectra has a very convenient universal property

A downside, at least from a pedagogical point of view, is that spectra are by nature 'derived' or 'homotopical' objects: not the individual cohomology groups H^n but rather the entire H^* takes center stage. Accordingly, it is really the ∞-*category* of spectra that has this convenient universal property. In algebraic geometry, such a 'derived' or 'homotopical' turn can be dated to the 80's when Beilinson suggested to construct a derived category of motives, something that was subsequently implemented independently by Hanamura, Levine and Voevodsky.

This entire strand of derived and homotopical motivic research in the last 40 years has been extremely rich and successful and these notes do not offer but an invitation to explore it. To wit, the goals and the outline are as follows:

- In Section 2 we trace the construction and features of the ∞-category of spectra mentioned above. This involves a brief introduction to the language of ∞-categories.
- In Section 3 the translation to algebraic geometry is discussed, resulting in the construction of the ∞-category of ($\mathbb{A}^1$-)motivic spectra. Part of this translation is recognizing what the attractive features of motivic spectra analogous to (i)–(iii) are.
- In Section 4 motivic spectra are put into action: some basic objects and maps between them are introduced, and we see how rich these are already. To connect with some of the other talks at the school, motivic refinements of enumerative geometry are treated too.
- We then turn in Section 5 to less elementary motivic spectra, that is, cohomology theories. We discuss in particular classical Weil cohomology theories, but also algebraic K-theory, and finally motivic cohomology. The latter provides the link with Voevodsky's derived category of motives and Chow motives which form the subject of another set of lectures at the school.

There are many other introductions to (stable) motivic homotopy theory. To mention just a few with a somewhat similar flavour: [Dég25; AE17; Bac21].

[1] Another take could be that in topology there are ordinary and generalized cohomology theories, and Morel–Voevodsky's impetus was to transport the generalized ones to algebraic geometry, with all their methods and tools.

Acknowledgments I thank the organizers of the school – Clément Dupont, Ulysse Mounoud, Nikola Tomić, Sofian Tur-Dorvault – for the opportunity to give this lecture series, and all participants of the winter school for making it such an engaging event.

I have tried to give attributions for the main results and I apologize for the instances where I failed to do that. I learned from many different people whose perspectives are reflected in these notes.

2 Topological preliminaries

The goal of this section is to make the definition of motivic spectra in the next section come about as naturally as possible. For this it behooves us to recall the construction and the basic features of spectra in topology. As mentioned in the introduction, this is preferrably phrased in the language of ∞-categories to which we offer a brief non-rigorous introduction. For a rigorous treatment see [Lur09; HA]. The shorter introduction [Gal] contains virtually everything we'll need but omits many proofs.

Spaces

Let $\mathcal{X}$ be a topological space. It has:

0. points
1. paths between points
2. homotopies between paths
3. homotopies between homotopies
4. etc ad infinitum

Moreover,

1. paths can be composed to form new paths, in fact in many different ways, and
2. any two choices of compositions are homotopic.
3. Not only that but such homotopies are themselves homotopic.
4. etc ad infinitum

We could summarize these last points by saying that composition of paths is *unique up to coherent homotopy*.

This resulting structure which encodes the homotopical information of $\mathcal{X}$ and nothing more goes under various names, such as the fundamental ∞-groupoid or the homotopy type of $\mathcal{X}$. When we forget $\mathcal{X}$ and just focus on this structure we call it a *space*. (Note how we dropped "topological".) Note that a space X has associated homotopy sets/groups

$$\pi_0(X) \in \mathsf{Set}, \qquad \pi_1(X,x) \in \mathsf{Grp}, \qquad \pi_n(X,x) \in \mathsf{Ab}\ (n \geq 2).$$

Moreover, we want to consider a map $f\colon X \to Y$ of spaces to be an equivalence if it induces bijections on all homotopy sets/groups.

Alternative 2.1. (If the above is not precise enough for you the following is at least a start.) One way of formalizing the notion of a space is via simplicial sets. Recall that the simplex category $\mathbf{\Delta}$ has objects the natural numbers $[n], n \geq 0$, and morphisms $[m] \to [n]$ are non-decreasing functions $\{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$. A simplicial set is a presheaf $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$.

Sending $[n]$ to the standard topological simplex $|\Delta^n| \subseteq \mathbb{R}^{n+1}$ extends to a functor $\mathbf{\Delta} \to \mathsf{Top}$ and to a topological space $\mathcal{X}$ we may associate the simplicial set

$$X := \mathrm{Hom}(|\Delta^\bullet|, \mathcal{X}),$$

called the singular simplicial set.

Exercise 2.2. Convince yourself that the singular simplicial set X does indeed capture the structure of the fundamental ∞-groupoid of $\mathcal{X}$ described above.

In the course of that you might want to look up what a *Kan complex* is and convince yourself that X is an example of such.

Remark 2.3. Let $x \rightsquigarrow y$ and $y \rightsquigarrow z$ be two paths in $\mathcal{X}$ parametrized by maps $f, g \colon |\Delta^1| \to \mathcal{X}$, that is, by $f, g \in X([1])$. Write $\underline{\mathrm{Hom}}(S, T) = \mathrm{Hom}(S \times \Delta^\bullet, T)$ for the internal hom in simplicial sets. The simplicial set of composites

$$\underline{\mathrm{Hom}}(\Delta^2, X) \times_{\underline{\mathrm{Hom}}(\Lambda^2_1, X)} \{(f, \bullet, g)\}$$

is a contractible (that is, its $\pi_i = *$ are all trivial) Kan complex. This makes precise the uniqueness up to coherent homotopy above.

∞-categories

Definition 2.4. An ∞-*category* has *objects* $c, d, \ldots$ and *mapping spaces* $\mathrm{Map}(c, d)$ for each pair (c, d). A *morphism* $f \colon c \to d$ is a point in $\mathrm{Map}(c, d)$. Morphisms can be composed uniquely up to coherent homotopy.

Example 2.5. Let X be a space. Then X is an ∞-category with objects its points, and with $\mathrm{Map}(x, y)$ the space of paths from x to y. (If X is the space associated to $\mathcal{X}$ then this is the space associated to $\{x\} \times_{\mathcal{X}} P\mathcal{X} \times_{\mathcal{X}} \{y\}$ where $P\mathcal{X}$ is the free path space.)

Example 2.6. Let C be an ordinary category. C can be viewed as an ∞-category with the same objects and discrete mapping spaces given by $\mathrm{Hom}(c, d)$. The composition is unique (not only up to coherent homotopy).

Construction 2.7. Let C be an ∞-category. Its *homotopy category* $\mathrm{ho}(C)$ has the same objects, and

$$\mathrm{Hom}_{\mathrm{ho}(C)}(c, d) = \pi_0 \, \mathrm{Map}(c, d) =: \mathrm{Hom}_C(c, d).$$

Since the composition of morphisms in C is associative up to (coherent) homotopy, $\mathrm{ho}(C)$ inherits the structure of an ordinary category.

Example 2.8. The homotopy category of a space is its fundamental 1-groupoid. The homotopy category of an ordinary category is just the ordinary category itself.

Definition 2.9. A morphism $f\colon c \to d$ in an ∞-category C is *invertible* (or an *equivalence*) if it is so in $\mathrm{ho}(C)$.

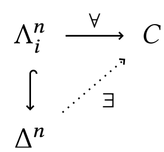
Exercise 2.10. Show that this is equivalent to the existence of $g\colon d \to c$ such that $fg \simeq \mathrm{id}$, $gf \simeq \mathrm{id}$.

Remark 2.11. Of course, in a space all morphisms are invertible (since a path can be run backwards). But in fact the converse holds as well. That is, for C an ∞-category the following are equivalent:

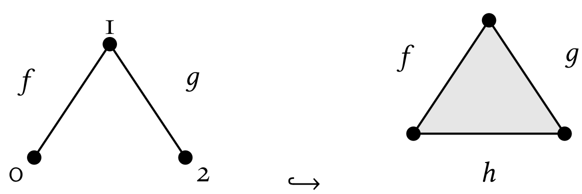
1. C is a space;
2. all morphisms in C are invertible.

Convention 2.12. If C is an ∞-category we denote by $C^{\simeq}$ its *core*: the (wide) subcategory spanned by invertible morphisms. In other words, this is the largest subspace of C.

Alternative 2.13. If you prefer a rigorous definition, an ∞-category is a simplicial set C such that for all $0 < i < n$:[2]



Here $\Lambda^n_i \subseteq \Delta^n := \mathrm{hom}(-,[n])$ is the 'ith inner horn', the subsimplicial set generated by all the faces except the ith. For example, for $n = 2$ (and necessarily $i = 1$) this is saying that for morphisms f and g where the domain of g and the codomain of f match, there is a composition $h \simeq g \circ f$:



Other choices of $n > 2$ and $0 < i < n$ express the fact that this composite is unique up to coherent homotopy.

Functors

Definition 2.14. A *functor* $F\colon C \to D$ of ∞-categories is a map $\mathrm{Ob}(C) \to \mathrm{Ob}(D)$ and maps of spaces $\mathrm{Map}_C(c,d) \to \mathrm{Map}_D(F(c),F(d))$ that are coherently compatible with composition.

Alternative 2.15. This is especially straightforward to express in the simplicial set model of ∞-categories: A functor $F\colon C \to D$ is simply a map of simplicial sets between ∞-categories!

Example 2.16. For C, D ordinary one recovers the usual notion of functors. For spaces one recovers maps of spaces.

[2] Specifically, this simplicial set model of ∞-categories is known as a quasicategory.

Definition 2.17. $F\colon C \to D$ is

- *fully faithful* if the maps of spaces $\mathrm{Map}(c, d) \to \mathrm{Map}(F(c), F(d))$ are equivalences;
- *essentially surjective* if $\mathrm{ho}(F)$ is;
- an *equivalence* if fully faithful and essentially surjective.

Remark 2.18. Given ∞-categories C, D the functors themselves assemble into an ∞-category $\mathrm{Fun}(C, D)$ with morphisms given by natural transformations $\{0 \to 1\} \times C \to D$.

Example 2.19. If C, D are ordinary then $\mathrm{Fun}(C, D)$ is the ordinary category of functors and natural transformations. If $\mathcal{X}, \mathcal{Y}$ are topological spaces then the functor ∞-category underlies the topological space of continuous maps $C \to D$ (with the compact-open topology).

Remark 2.20. There exists a (large) ∞-category Cat_∞ with objects (small) ∞-categories and $\mathrm{Map}(C, D) = \mathrm{Fun}(C, D)^{\simeq}$. From the preceding discussion we have two full subcategories:

- Spc: spaces $X, Y, \dots$ and mapping spaces $\mathrm{Map}(X, Y) = \mathrm{Fun}(X, Y)$ (since $\mathrm{Fun}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$).
- Cat_1: ordinary categories, functors and *invertible* natural transformations.

Example 2.21. It is a fact that a functor $F\colon C \to D$ of ∞-categories is an equivalence in the sense of Definition 2.17 iff it is an equivalence in Cat_∞ in the sense of Definition 2.9.

Adjunctions

Many further concepts from ordinary categories can now easily be translated to ∞-categories. For example:

Definition 2.22. An *adjunction* $F\colon C \rightleftarrows D :G$ consists of natural transformations $\mathrm{id} \to GF$ and $FG \to \mathrm{id}$ with the usual triangle identities witnessed by homotopies.

Definition 2.23. Let $F\colon K \to C$ be a functor. A *colimit* of F is an object $c \in C$ together with a natural transformation $F \to \Delta(c)$ of functors (with, of course, $\Delta(c)\colon K \to * \xrightarrow{c} C$ the constant functor) such that the composite

$$\mathrm{Map}(c, d) \xrightarrow{\Delta} \mathrm{Map}(\Delta(c), \Delta(d)) \to \mathrm{Map}(F, \Delta(d))$$

is an equivalence of spaces for all $d \in C$.

Remark 2.24. The expected properties hold:

- These notions recover the usual ones for ordinary categories.
- Left adjoints preserve colimits.
- The space of colimits (resp. left adjoints) is either empty or contractible.
- Limits are defined in a similar way, and the dual statements are true.

Example 2.25. 1. The inclusion $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}_\infty$ admits a left adjoint given by the homotopy category $\mathrm{ho}\colon \mathsf{Cat}_\infty \to \mathsf{Cat}_1$.

2. The inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ admits a right adjoint given by the core $(-)^\simeq\colon \mathsf{Cat}_\infty \to \mathsf{Spc}$.

3. $c \in C$ is initial iff $\mathrm{Map}(c, d) \simeq *$ for all $d \in C$.

Remark 2.26. The first item is an example of a *localization*: a left adjoint with fully faithful right adjoint.

Alternative 2.27. In ordinary category theory the Yoneda lemma 'reduces the computation' of (co)limits in categories to those in sets. Similarly, (co)limits in ∞-categories can be 'tested' in Spc. In that case, they coincide with the classical notion of *homotopy limits*. To give an idea, the homotopy pullback of topological spaces

$$\begin{array}{ccc} \mathcal{X} \times^h_{\mathcal{Z}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \end{array}$$

has points $x \in \mathcal{X}$, $y \in \mathcal{Y}$ together with a path $f(x) \rightsquigarrow g(y)$ in $\mathcal{Z}$. (To be precise, it is the ordinary pullback $\mathcal{X} \times_{\mathcal{Z}} P\mathcal{Z} \times_{\mathcal{Z}} \mathcal{Y}$.) Since homotopy products are just ordinary products, this gives a recipe in general for how to compute homotopy limits in topological spaces.

Presentability

Remark 2.28. Because it is virtually impossible to write down functors by hand, the adjoint functor theorem is more important for ∞-categories than for ordinary categories. Presentable ∞-categories admit a very convenient adjoint functor theorem. This notion is a bit technical but the upshot is that a presentable ∞-category is complete and cocomplete and determined by a small ∞-category. An important example is a presheaf category.

Definition 2.29. Let C be a small ∞-category. The *presheaf category* $\mathcal{P}(C)$ is the functor category $\mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Spc})$. It comes with the *Yoneda functor* $\mathrm{y}\colon C \to \mathcal{P}(C)$ which is fully faithful.

Remark 2.30. The presheaf category is the *free cocompletion*: it is cocomplete, and for every cocomplete D, restriction along Yoneda induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

where the left-hand side denotes the full subcategory on colimit preserving functors.

Example 2.31. The ∞-category of spaces is the free cocompletion on a point:

$$\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Spc}, D) \xrightarrow{\sim} D$$

by evaluation at the final object $*$.

Definition 2.32. Let κ be a regular cardinal. An ∞-category L is called *κ-filtered* if for every κ-small[3] ∞-category K and every $F\colon K \to L$ there exists an 'upper bound': an object $\ell \in L$ and a natural transformation $F \to \Delta(\ell)$. Finally, let C be a small ∞-category. We denote by $\mathrm{Ind}_\kappa(C) \subseteq \mathcal{P}(C)$ the full subcategory on presheaves that are κ-filtered colimits of representables.

A *presentable* ∞-category is one of the form $\mathrm{Ind}_\kappa(C)$ for some small C and for some regular cardinal κ.

Remark 2.33. Equivalently, a presentable ∞-category is an accessible localization of $\mathcal{P}(C)$ for some small C, that is, the (fully faithful) right adjoint is *accessible*, meaning: preserves κ-filtered colimits for some regular κ.

Example 2.34. Cat_1, Spc, Cat_∞ are all presentable (with $\kappa = \aleph_0$).

We now come to the adjoint functor theorem for presentable ∞-categories. A simplified version that will be enough for us says:

Remark 2.35. A functor $F\colon C \to D$ between presentable ∞-categories is

- left adjoint iff it preserves colimits:
- right adjoint iff it preserves limits and is accessible.

Convention 2.36. We denote by $\mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{CAT}_\infty$ the subcategory (of large ∞-categories) on presentable ∞-categories and left adjoint functors. Similarly, $\mathrm{Pr}^{\mathrm{R}} \subseteq \mathrm{CAT}_\infty$ are the presentable ∞-categories and right adjoint functors.

Remark 2.37. Passing to adjoints induces an equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$.

Remark 2.38. Presentable ∞-categories are stable under many constructions. For example, both forgetful functors $\mathrm{Pr}^{\mathrm{L}}, \mathrm{Pr}^{\mathrm{R}} \to \mathrm{CAT}_\infty$ preserve limits. Together with the previous remark this means that colimits in Pr^{L} are computed by passing to right adjoints and computing the limit in CAT_∞.

Stability

An area where ∞-categories shine is homological and, more generally, higher algebra. If you're familiar with triangulated categories (e.g. derived categories), *stable* ∞-categories enhance these in the sense that their homotopy categories have a canonical triangulated structure.[4]

Definition 2.39. An ∞-category is *pointed* if it has initial objects and final objects and these coincide.

[3]This means that objects and all homotopy groups of all mapping spaces are κ-small.
[4]Rather, and more accurately, there are *two* such structures, one being the negative of the other.

Example 2.40. The ∞-category of pointed spaces $\mathsf{Spc}_* := \mathsf{Spc}_{*/}$[5] is pointed. In fact, adding a disjoint base point $(\ )_+ \colon \mathsf{Spc} \to \mathsf{Spc}_*$ is the initial pointed ∞-category in $\mathrm{Pr}^{\mathrm{L}}_{\mathsf{Spc}/-}$. Alternatively, it is the free pointed cocompletion on a point: for any cocomplete pointed D,

$$\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Spc}_*, D) \xrightarrow{\sim} D$$

by evaluation at $S^0 := (*)_+$.

Definition 2.41. Let C be pointed and have finite limits and colimits. Given a map $f \colon c \to d$ we define *fiber* (= 'homotopy kernel') and *cofiber* (= 'homotopy cokernel') as

$$\mathrm{fib}(f) := \lim(c \xrightarrow{f} d \leftarrow *), \qquad \mathrm{cof}(f) := \mathrm{colim}(* \leftarrow c \xrightarrow{f} d).$$

In particular, we have *suspension* and *loop* functors, with $\Sigma(c) := \mathrm{cof}(c \to *)$ and $\Omega(c) := \mathrm{fib}(* \to c)$.

Σ is left adjoint to Ω.

Definition 2.42. A pointed ∞-category is *stable* if it has finite limits and colimits, and a square is cartesian iff it is cocartesian. Functors are called *exact* if they preserve finite limits and colimits.

Remark 2.43. It turns out that C is stable iff $\Sigma \dashv \Omega$ are inverse equivalences to each other.

Remark 2.44. Let C, D be stable. A functor $F \colon C \to D$ is exact iff it preserves finite limits iff it preserves finite colimits.

Convention 2.45. We denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ the full subcategory on stable presentable ∞-categories. Observe that the morphisms in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ are automatically exact functors.

Definition 2.46. The ∞-category of *spectra* Sp is the initial stable presentable ∞-category $\Sigma^\infty \colon \mathsf{Spc}_* \to \mathsf{Sp}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathsf{Spc}_*/}$. Alternatively, it is the free stable cocompletion on a point: for any stable cocomplete D,

$$\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Sp}, D) \xrightarrow{\sim} D$$

by evaluation at the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$.

Remark 2.47. Remark 2.43 suggests how to construct Sp from Spc_*: it is obtained by inverting Σ, that is, Sp is the colimit in Pr^{L} of the diagram

$$\cdots \to \mathsf{Spc}_* \xrightarrow{\Sigma} \mathsf{Spc}_* \xrightarrow{\Sigma} \cdots .$$

By Remark 2.38, it can also be viewed as the limit in CAT_∞ of the diagram

$$\cdots \leftarrow \mathsf{Spc}_* \xleftarrow{\Omega} \mathsf{Spc}_* \xleftarrow{\Omega} \cdots ,$$

Hence a spectrum is a collection of pointed spaces E_n and equivalences $E_n \simeq \Omega E_{n+1}$.

[5] You may define the slice category $C_{c/}$ as the pullback of the diagram $\mathrm{Fun}(\{0 \to 1\}, C) \xrightarrow{0^*} C \leftarrow \{c\}$.

Monoidal structures

Traditionally, a symmetric monoidal category is an ordinary category C and a functor $\otimes\colon C \times C \to C$ together with various natural isomorphisms (associativity, symmetry, unitality) satisfying certain properties. One way to encode the necessary higher coherences in the case of an ∞-category C is as follows.

Convention 2.48. We denote by Fin_* the ordinary category of finite pointed sets. Note that every object is isomorphic to one of the form $\langle n \rangle := \{*, 1, \dots, n\}$ for $n \geq 0$. We denote by $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$ the map that sends all but i to the base point $*$.

Definition 2.49. A *symmetric monoidal ∞-category* is a functor $\mathscr{C}\colon \mathrm{Fin}_* \to \mathsf{Cat}_\infty$ such that for each $n \geq 0$, the ρ_i (for $i = 1, \dots, n$) induce an equivalence

$$\mathscr{C}_{\langle n \rangle} \xrightarrow{\sim} (\mathscr{C}_{\langle 1 \rangle})^{\times n}.$$

The ∞-category $\mathscr{C}_{\langle 1 \rangle}$ is called the *underlying ∞-category*. We often abusively say that $C \in \mathsf{Cat}_\infty$ is symmetric monoidal without specifying the structure with underlying ∞-category C.

A *symmetric monoidal functor* is simply a natural transformation between symmetric monoidal ∞-categories.

Remark 2.50. Let $\mathscr{C}$ be a symmetric monoidal ∞-category and denote by $C = \mathscr{C}_{\langle 1 \rangle}$ the underlying ∞-category. The map $\langle 2 \rangle \to \langle 1 \rangle$ that sends both 1 and 2 to 1 is taken by $\mathscr{C}$ to a functor $\otimes\colon C \times C \to C$ which we interpret as the symmetric monoidal (bi)product. The value of $\mathscr{C}$ on the unique map $\langle 0 \rangle \to \langle 1 \rangle$ is a functor $* \to C$ that picks out a unit for $\otimes$.

Exercise 2.51. Extract the associativity and symmetry constraints from $\mathscr{C}$.

Example 2.52. When C has finite products then it admits a symmetric monoidal structure for which the product is the categorical product. This is called the cartesian monoidal structure. For example this applies to Spc and Cat_∞.

Remark 2.53. Given a symmetric monoidal ∞-category C one can speak about $\mathrm{CAlg}(C)$, commutative algebras in C. We will not define these since they require the language of ∞-operads. In the case of a cartesian monoidal structure it can be done just as in Definition 2.49. For example, a commutative algebra in Cat_∞ (with the cartesian monoidal structure) is nothing but a symmetric monoidal ∞-category.

Using the symmetric monoidal structures we can now give alternative universal properties of Spc, Spc_*, Sp.

Example 2.54. Spc is the initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. In fact, for any $D \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ we have

$$\mathrm{Fun}^{L,\otimes}(\mathsf{Spc}, D) \simeq *.$$

Example 2.55. The ∞-category of pointed spaces Spc_* has a unique symmetric monoidal structure $\wedge$ compatible with colimits such that adding a disjoint base point $(-)_+\colon \mathsf{Spc} \to \mathsf{Spc}_*$ is symmetric monoidal. It is called the smash product, and $X \wedge Y \simeq X \times Y / X \vee Y$. The unit is S^0. Just as before, Spc_* is the initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_*)$ (where $\mathrm{Pr}^{\mathrm{L}}_*$ are, of course, pointed presentable ∞-categories).

Remark 2.56. The ∞-category of spectra has a unique symmetric monoidal structure compatible with colimits such that $\Sigma^\infty \colon \mathsf{Spc}_* \to \mathsf{Sp}$ is symmetric monoidal. In fact, it is the initial object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ and one even has for any $D \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$:

$$\mathrm{Fun}^{L,\otimes}(\mathsf{Sp}, D) \simeq *$$

We continute to denote this symmetric monoidal structure on Sp by $\wedge$. The unit is $\mathbb{S} = \Sigma^\infty S^0$.

Exercise 2.57. Show that as endofunctors of Spc_*, $\Sigma \simeq \wedge S^1$. It follows that $\Sigma^\infty \colon \mathsf{Spc}_* \to \mathsf{Sp}$ is also the initial way in $\mathrm{Pr}^{\mathrm{L}}_*$ to invert S^1 (that is, to make $\wedge S^1$ an autoequivalence).

Cohomology theories

We finally link spectra with the notion of a cohomology theory for topological spaces.

Remark 2.58. Recall that a cohomology theory in topology is a collection of functors

$$H^n \colon \mathrm{CW}_*^{\mathrm{op}} \to \mathsf{Set}, \qquad n \in \mathbb{Z},$$

on pointed CW complexes[6] together with isomorphisms ('bonding maps') $\gamma \colon H^{n+1}\Sigma \cong H^n$ satisfying:

- *homotopy invariance*: $H^n(X) \xrightarrow{\sim} H^n(X \times [0,1])$
- *sheaf conditions*:
  - For any $X_i \in \mathrm{CW}_*$, $H^n(\bigvee X_i) \xrightarrow{\sim} \prod H^n(X_i)$.
  - H^n sends (homotopy) pushouts in CW_* to 'weak' pullbacks, meaning the canonical map to the pullback is surjective.

Remark 2.59. Brown representability implies that each H^n is representable by a pointed CW-complex E_n (unique up to homotopy equivalence) and $E_n \simeq \Omega E_{n+1}$. In particular, it gives rise to a spectrum $E = (E_n)$ (Remark 2.47). Conversely, a spectrum $E = (E_n) \in \mathsf{Sp}$ yields a cohomology theory via $H^n = \mathrm{Hom}_{\mathsf{Spc}_*}(-, E_n)$ and the isomorphisms

$$H^{n+1}\Sigma = \mathrm{Hom}(\Sigma(-), E_{n+1}) = \mathrm{Hom}(-, \Omega E_{n+1}) = \mathrm{Hom}(-, E_n) = H^n.$$

Altogether these are part of a bijection

$$\text{cohomology theories}/\simeq \quad \leftrightarrow \quad \text{spectra}/\simeq$$

Example 2.60. Let A be an abelian group. The singular cohomology $X \mapsto \mathrm{H}^*(X; A)$ is represented by a spectrum $\mathrm{H}A \in \mathsf{Sp}$, called the Eilenberg–MacLane spectrum associated with A. In degrees $n \geq 0$ it is the Eilenberg–MacLane space $K(A, n)$ with canonical homotopy equivalences $\Omega K(A, n+1) \simeq K(A, n)$, see Exercise 2.61.

[6] It is enough to consider *finite* CW complexes.

Exercise 2.61. 1. Assume the statements in Example 2.60 and compute the homotopy groups of $K(A,n)$. The latter is characterized by being a CW complex and having these homotopy groups. Deduce that indeed $\Omega K(A,n+1) \simeq K(A,n)$ canonically.

2. Conversely, with this definition of $K(A,n)$ show that $\mathrm{H}A = (K(A,n))_{n\geq 0}$. (You might want to use the characterization of singular cohomology due to Eilenberg–Steenrod.)

Example 2.62. Let $KU \in \mathsf{Sp}$ represent complex topological K-theory. It has pointed spaces

$$BU \times \mathbb{Z},\ \Omega BU,\ BU \times \mathbb{Z}, \ldots$$

where BU is the classifying space for the infinite unitary group and Bott periodicity $\Omega^2 BU \simeq \mathbb{Z} \times BU$ as the non-trivial bonding maps.

Exercise 2.63. The *stable homotopy groups* of a spectrum $E \in \mathsf{Sp}$ are defined as

$$\pi_m(E) := \mathrm{Hom}(\mathbb{S}^m, E), \qquad m \in \mathbb{Z}.$$

If $E = (E_n)$ with $\Omega E_{n+1} \simeq E_n$ you can compute these as

$$\pi_m(E) = \pi_{m+n} E_n$$

for any n such that $m + n \geq 0$. Compute the homotopy groups of

1. $\mathrm{H}A$ (this is easy if you did the previous exercise),
2. KU (the even ones are easy and I'll just tell you that the odd ones vanish).

Remark 2.64. In summary we have now seen several perspectives on spectra:

1. The ∞-category Sp arises as

$$* \rightsquigarrow \mathsf{Spc} \rightsquigarrow \mathsf{Spc}_* \rightsquigarrow \mathsf{Sp}$$

by free cocompletion followed by rendering it pointed and stable. In particular, Sp is the free stable cocompletion on a point.

2. $(\mathsf{Sp}, \wedge)$ is the initial object of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$.
3. An object of Sp is a collection of pointed spaces E_n together with equivalences $E_n \simeq \Omega E_{n+1}$.
4. An object of Sp is a cohomology theory on pointed CW complexes.

3 Definition

Convention 3.1. Let k be a field. All schemes are separated and of finite type over k.

In this section we will give the formal definition of motivic spectra Sp_k following the template of 1 in Remark 2.64: we first define motivic spaces Spc_k, pass to pointed objects, and then ($\mathbb{P}^1$-)stabilize. We will also have analogues of the other perspectives in Remark 2.64. Let us anticipate the outcome by describing what a cohomology theory represented by a motivic spectrum will look like. You will no doubt see the parallel to Remark 2.58. The differences will be discussed subsequently. (Some of the terms will not necessarily make sense to you yet. They will all be introduced below.)

Definition 3.2. A *Morel–Voevodsky cohomology theory* is a collection of functors

$$H^n\colon (\mathrm{Sm}_k)^{\mathrm{op}} \to \mathsf{Spc}_*, \qquad n \in \mathbb{Z},$$

on smooth k-schemes together with equivalences $\gamma\colon \Omega_{\mathbb{P}^1} H^{n+1} \simeq H^n$ satisfying:

- *homotopy invariance*: $H^n(X) \xrightarrow{\sim} H^n(X \times \mathbb{A}^1)$
- *sheaf conditions*:
  - $H^n(\emptyset) = *$
  - H^n sends Nisnevich squares to pullbacks.

3.1 Motivic spaces

Let us talk about the differences between Definition 3.2 and Remark 2.58.

Remark 3.3. The interval is replaced by the affine line $\mathbb{A}^1$. This is quite natural, of course, and gives the theory its name.

Remark 3.4. Just as there are technical reasons to choose CW complexes over all topological spaces, our algebro-geometric cohomology theories are not defined on all schemes but only on smooth schemes. Closely related to that, the sheaf condition is with respect to the Nisnevich topology. We'll come back to these choices in Remarks 3.31 and 3.32.

Definition 3.5. For now, let us define a *(distinguished, or elementary) Nisnevich square* to be a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & X \end{array} \tag{3.6}$$

where j is an open immersion, f is étale and induces an isomorphism $(V \backslash W)_{\mathrm{red}} \xrightarrow{\sim} (X \backslash U)_{\mathrm{red}}$.

Example 3.7. If j and f are both open immersions and $U \cup V = X$ then the resulting cartesian square is a Nisnevich square. (And also a Zariski cover.)

Exercise 3.8. Let $\mathrm{char}(k) \neq 2$ and $\lambda \in k^\times$. Let $j\colon \mathbb{A}^1 - \{\lambda^2\} \hookrightarrow \mathbb{A}^1$ and $f\colon \mathbb{A}^1 - \{0, \lambda\} \to \mathbb{A}^1$ the "squaring" map $x \mapsto x^2$. Show that these two maps induce a Nisnevich square (which is not a Zariski cover).

Remark 3.9. The change from Set-valued presheaves in Remark 2.58 to Spc_*-valued ones in Definition 3.2 is less of a conceptual leap than might appear. Indeed, you can easily convince yourself that the H^n in topology uniquely lift to Set_* (because of the base point in CW_*), and given the Brown representability result, also to Spc_*.[7]

On the other hand, we do not know of a characterization of MV-cohomology theories in terms of presheaves valued just in sets (except if one changes smooth schemes on which the presheaves are defined to something substantially richer). This makes the construction of such cohomology theories apriori more involved than in topology.

Definition 3.10. An *($\mathbb{A}^1$-)motivic space* is a functor $F\colon (\mathrm{Sm}_k)^{\mathrm{op}} \to \mathsf{Spc}$ satisfying:

- *homotopy invariance*: $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for all $X \in \mathrm{Sm}_k$;
- *Nisnevich excision*: F takes a Nisnevich square (3.6) with X smooth to a cartesian square in spaces and $F(\emptyset) \simeq *$.

Motivic spaces form a full subcategory $\mathsf{Spc}_k \subseteq \mathcal{P}(\mathrm{Sm}_k)$.

Remark 3.11.
1. Of course, $\mathcal{P}(\mathrm{Sm}_k)$ is a presentable ∞-category. The inclusion $\mathsf{Spc}_k \hookrightarrow \mathcal{P}(\mathrm{Sm}_k)$ is accessible and admits a left adjoint $L_{\mathbb{A}^1}$ ("motivic localization") so that Spc_k is also presentable.
2. $\mathcal{P}(\mathrm{Sm}_k)$ has a unique ('pointwise') symmetric monoidal structure compatible with colimits so that the Yoneda embedding is symmetric monoidal.

   The motivic localization is compatible with the symmetric monoidal structure and therefore endows Spc_k with a symmetric monoidal structure $\wedge$ compatible with colimits. Alternatively, there is a unique lift of $L_{\mathbb{A}^1}\colon \mathcal{P}(\mathrm{Sm}_k) \to \mathsf{Spc}_k$ to an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathcal{P}(\mathrm{Sm}_k)/}$.

Convention 3.12. In particular, the Yoneda embedding followed by motivic localization induce a symmetric monoidal functor

$$\mathrm{Sm}_k \xrightarrow{\mathrm{y}} \mathcal{P}(\mathrm{Sm}_k) \xrightarrow{L_{\mathbb{A}^1}} \mathsf{Spc}_k$$

and we often abusively write X for the image of $X \in \mathrm{Sm}_k$ under this composite.

Exercise 3.13. Given a Nisnevich square (3.6), show that its image in Spc_k is a cocartesian square.

Exercise 3.14. Let $p\colon E \to X$ be a vector bundle. Then p induces an equivalence in Spc_k.

Remark 3.15. There is also a unique lift of $(\ )_+\colon \mathsf{Spc}_k \to (\mathsf{Spc}_k)_*$ to $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})_{\mathsf{Sp}_k/}$. This symmetric monoidal structure on $(\mathsf{Sp}_k)_*$ is denoted by $\wedge$ as in topology.

3.2 Stabilization

Remark 3.16. In topology, the connection between the different H^n was expressed in terms of suspension, that is, the smash product with S^1 in pointed spaces (Exercise 2.57). Passing to

[7] In fact, even to infinite loop spaces.

spectra can be viewed as a process of inverting S^1. In algebraic geometry, there are several spheres available. For example, the $\mathbb{R}$- and $\mathbb{C}$-points of $\mathbb{G}_m$ and $\mathbb{P}^1$, respectively, are

$$\mathbb{G}_m(\mathbb{R}) \simeq S^0, \quad \mathbb{G}_m(\mathbb{C}) \simeq S^1, \quad \mathbb{P}^1(\mathbb{R}) \simeq S^1, \quad \mathbb{P}^1(\mathbb{C}) \simeq S^2.$$

It will turn out that inverting $\mathbb{P}^1$ in our context will automatically make $\mathbb{G}_m$ and S^1 invertible too (see Example 4.2). In particular, motivic spectra will automatically be stable.

Definition 3.17 (Morel–Voevodsky [MV99], Robalo [Rob15])**.** There is an initial recipient $\Sigma^\infty\colon (\mathsf{Spc}_k)_* \to \mathsf{Sp}_k$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ inverting $(\mathbb{P}^1, \infty)$. This is called the ∞-category of ($\mathbb{A}^1$-)*motivic spectra* or *stable* $\mathbb{A}^1$*-homotopy theory*.

Remark 3.18. In summary, our definition of Sp_k follows the template of 1 in Remark 2.64:

$$\mathsf{Spc}_k \rightsquigarrow (\mathsf{Spc}_k)_* \rightsquigarrow \mathsf{Sp}_k$$

by rendering motivic spaces pointed and then inverting $\mathbb{P}^1$.

Convention 3.19.

1. If the base point is clear from the context or irrelevant we omit it from the notation.
2. We often omit the infinite suspension Σ^∞ from the notation as well.

Hence we write $\mathbb{P}^1$ for all of the following three objects:

$$\mathbb{P}^1 \in \mathsf{Spc}_k, \quad (\mathbb{P}^1, \infty) \in (\mathsf{Spc}_k)_*, \quad \Sigma^\infty(\mathbb{P}^1, \infty) \in \mathsf{Sp}_k$$

Remark 3.20. One can give a more explicit description of Sp_k. (This relies on the fact that the cylic permutation acts identically on $(\mathbb{P}^1)^{\wedge 3}$.) Namely, it is the sequential colimit of

$$\cdots \to (\mathsf{Spc}_k)_* \xrightarrow{\wedge \mathbb{P}^1} (\mathsf{Spc}_k)_* \xrightarrow{\wedge \mathbb{P}^1} \cdots$$

in Pr^{L}. Equivalently (Remark 2.38), it is the sequential limit of

$$\cdots \leftarrow (\mathsf{Spc}_k)_* \xleftarrow{\Omega_{\mathbb{P}^1}} (\mathsf{Spc}_k)_* \xleftarrow{\Omega_{\mathbb{P}^1}} \cdots$$

where we denote by $\Omega_{\mathbb{P}^1}$ the right adjoint functor $\underline{\mathrm{Hom}}(\mathbb{P}^1, -)$. In particular, we deduce an analogue of 3 and 4 in Remark 2.64:

Corollary 3.21. *A motivic spectrum $E = (E_n) \in \mathsf{Sp}_k$ is the same thing as a Morel–Voevodsky cohomology theory (Definition 3.2).*

Exercise 3.22. Show that the functor $\Omega_{\mathbb{P}^1}\colon (\mathsf{Spc}_k)_* \to (\mathsf{Spc}_k)_*$ is given by

$$\Omega_{\mathbb{P}^1} F(X) = \mathrm{fib}\left(F(X \times \mathbb{P}^1) \to F(X \times \{\infty\})\right).$$

Finally, we give a universal property of Sp_k in the spirit of Remark 2.64.2.

Corollary 3.23. *For any $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$, the canonical functor $(\ )_+ \colon \mathrm{Sm}_k \to \mathsf{Sp}_k$ induces a fully faithful embedding*

$$\mathrm{Fun}^{L,\otimes}(\mathsf{Sp}_k, C) \to \mathrm{Fun}^{\otimes}(\mathrm{Sm}_k, C)$$

onto those functors F satisfying:

1. *F sends the canonical projection to $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$,*
2. *F sends a Nis-square (3.6) to a cocartesian square and $F(\emptyset) \simeq 0$,*
3. *$F(\mathbb{P}^1)$ is invertible.*

Proof. We can factor the functor as

$$\mathrm{Fun}^{L,\otimes}(\mathsf{Sp}_k, C) \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}} \mathrm{Fun}^{L,\otimes}((\mathsf{Spc}_k)_*, C) \xrightarrow{(\ )_+} \mathrm{Fun}^{L,\otimes}(\mathsf{Spc}_k, C) \xrightarrow{L_{\mathbb{A}^1}} \mathrm{Fun}^{L,\otimes}(\mathscr{P}(\mathrm{Sm}_k), C) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(\mathrm{Sm}_k, C)$$

and each of these is fully faithful by the respective universal properties of the symmetric monoidal functors stated before. The description of the image is clear. □

Example 3.24. This universal property makes it relatively easy to construct realizations. For example, assume $k = \mathbb{C}$ and consider the functor $(\mathbb{C}) \colon \mathrm{Sm}_{\mathbb{C}} \to \mathsf{Sp}$ which sends X to $X(\mathbb{C})$ with the analytic topology. Since $\Sigma^\infty_+ \colon \mathsf{Spc} \to \mathsf{Sp}$ is symmetric monoidal so is $\mathfrak{R}_{\mathbb{C}} = \Sigma^\infty_+ \circ (\mathbb{C})$. We now observe:

1. $\mathbb{A}^1(\mathbb{C}) = \mathbb{C} \simeq *$ thus homotopy invariance.
2. One can show that the analytification of a Nis-square is cocartesian in Spc. (This is essentially because the analytification of an étale map is a local homeomorphism.) Since Σ^∞_+ preserves colimits, the same is true in Sp.
3. As already remarked, $\mathbb{P}^1(\mathbb{C}) = S^2$. We deduce that $\mathfrak{R}_{\mathbb{C}}(\mathbb{P}^1) = \mathbb{S}^2$ which is invertible.

Altogether we deduce from Corollary 3.23 a symmetric monoidal colimit preserving functor $\mathfrak{R}_{\mathbb{C}} \colon \mathsf{Sp}_{\mathbb{C}} \to \mathsf{Sp}$, called the *complex realization*.

Remark 3.25. One can further compose with $\mathrm{H}\mathbb{Z}\wedge \colon \mathsf{Sp} \to \mathrm{Mod}_{\mathrm{H}\mathbb{Z}} \simeq D(\mathbb{Z})$ to get Betti homology.

Example 3.26. Similarly, there is a *real realization* $\mathfrak{R}_{\mathbb{R}} \colon \mathsf{Sp}_{\mathbb{R}} \to \mathsf{Sp}$ induced by sending $X \in \mathrm{Sm}_{\mathbb{R}}$ to $\Sigma^\infty_+ X(\mathbb{R})$ with the analytic topology.

Exercise 3.27. If you're familiar with basic equivariant homotopy theory construct a C_2-equivariant realization $\mathfrak{R}_{C_2} \colon \mathsf{Sp}_{\mathbb{R}} \to \mathsf{Sp}_{C_2}$.

3.3 Motivic sheaves

Remark 3.28. To a smooth k-scheme we have associated an object in Sp_k which we can think of as its stable motivic homotopy type. Extending this to singular schemes involves the six-functor formalism which we briefly introduce.

Remark 3.29 (Ayoub [Ay007], Cisinski–Déglise [CD19])**.**

1. Nothing in the construction of Sp_k relied on k being a field. Indeed, for any scheme S we may start with Sm_S and consider the subcategory $\mathsf{Spc}_S \subseteq \mathcal{P}(\mathrm{Sm}_S)$ of $\mathbb{A}^1$-invariant Nisnevich sheaves. Sp_S is the $\mathbb{P}^1$-stabilization as before.
2. For a morphism $f\colon S \to T$, the functor
$$\mathrm{Sm}_T \xrightarrow{-\times_T S} \mathrm{Sm}_S \to \mathsf{Sp}_S$$
clearly satisfies the analogous conditions of Corollary 3.23 thus a symmetric monoidal left adjoint
$$f^*\colon \mathsf{Sp}_T \to \mathsf{Sp}_S$$
called *pullback*. Its right adjoint is denoted f_* and called *push-forward*.
3. There is also an adjunction $f_!\colon \mathsf{Sp}_S \rightleftarrows \mathsf{Sp}_T : f^!$ between *exceptional* pushforward and pullback. The former can be computed as follows. By Nagata compactification (and our scheme conventions) each morphism factors as $f = p \circ j$ where p is proper and j an open immersion. The functor j^* has a left adjoint $j_!$ which is induced by $\mathrm{Sm}_S \to \mathrm{Sm}_T$, $X \mapsto X$. In this situation, $f_! \simeq p_* \circ j_!$. (Once one knows that p_* commutes with colimits, the adjoint functor theorem (Remark 2.35) guarantees the existence of $f^!$.)

Convention 3.30. Let $f\colon X \to \mathrm{Spec}(k)$ be a k-scheme. The association $f \mapsto f_! f^! \mathbb{S}$ underlies a functor $M\colon\ \mathrm{Sch}_k \to \mathsf{Sp}_k$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Sm}_k & \xrightarrow{\Sigma^\infty_{\mathbb{P}^1,+} L_{\mathbb{A}^1}\, \mathrm{y}} & \mathsf{Sp}_k \\ \big\downarrow & \nearrow_{M} & \\ \mathrm{Sch}_k & & \end{array}$$

The object $M(X) \in \mathsf{Sp}_k$ is the *(homological) motive* associated with $X \in \mathrm{Sch}_k$.

Remark 3.31. These four functors $f^*, f_*, f_!, f^!$ together with $\wedge$ and internal hom satisfy many compatibilities, the structure of which is best encoded in terms of categories of correspondences. We will not develop this here but mention that such data goes by the name of *six-functor formalisms*. Examples are, besides motivic spectra, constructible étale derived categories, holonomic $\mathcal{D}$-modules, mixed Hodge modules and many more. There is a meta-theorem that motivic spectra form the initial six-functor formalism satisfying some natural axioms, see for instance [DG22]. We'd like to draw attention to two points:

- The axioms make no mention of the Nisnevich topology at all so that this result can be seen as an aposteriori justification for choosing this topology.
- If a cohomology theory is supposed to entail a six-functor formalism – this hypothesis is certainly in line with beliefs held by early proponents of a theory of motives – then this result can be seen as substantiating motivid spectra's claim to being a 'universal cohomology theory'.

Remark 3.32. One of the natural axioms alluded to above is *localization*. Given $X \in \mathrm{Sch}_k$ and a closed immersion $i\colon Z \hookrightarrow X$ with open complement $j\colon U \hookrightarrow X$, the ∞-category Sp_X is 'glued' from Sp_Z and Sp_U in a precise way. A consequence is the 'glueing theorem' [MV99], which says that

$$j_! j^* \to \mathrm{id} \to i_! i^* \tag{3.33}$$

is a bifiber sequence of endofunctors of Sp_X. The proof of this result seems to rely heavily on the choice of smooth schemes and the Nisnevich topology.

Remark 3.34. Given $p\colon X \to \mathrm{Spec}(k)$ in Sch_k the object $M^c(X) := p_!\mathbb{S} \in \mathsf{Sp}_k$ is compact. (That is, $\mathrm{Map}(p_!\mathbb{S}, -)\colon \mathsf{Sp}_k \to \mathsf{Spc}$ commutes with filtered colimits.) In contrast to Convention 3.30, $M^c(X)$ is the *cohomological motive with compact support* associated with X. Evaluating (3.33) on $\mathbb{S}$ and applying $p_!$ to it we get a bifiber sequence

$$M^c(U) \to M^c(X) \to M^c(Z)$$

in $(\mathsf{Sp}_k)^\omega$. It also follows from the properties of the six functors that

$$M^c(X) \otimes M^c(Y) \simeq M^c(X \times Y).$$

Together this yields a ring homomorphism

$$[M^c]\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0((\mathsf{Sp}_k)^\omega),$$

where the latter is the free abelian group on isomorphism classes of objects in $(\mathsf{Sp}_k)^\omega$ modulo the relation $A + C = B$ for any bifiber sequence $A \to B \to C$.

Most other motivic measures will factor through this one. For example, composing with the complex realization yields a ring homomorphism

$$[\mathfrak{R}_\mathbb{C} \circ M^c]\colon \mathrm{K}_0(\mathrm{Var}_\mathbb{C}) \to \mathrm{K}_0((\mathsf{Sp}_\mathbb{C})^\omega) \to \mathrm{K}_0(\mathsf{Sp}^\omega) \simeq \mathbb{Z}$$

which is nothing but the topological Euler characteristic.

4 Basic objects and maps

4.1 Bigraded spheres

Convention 4.1. By the universal property of Spc_* (Example 2.54) there exists a unique colimit preserving symmetric monoidal functor $\mathsf{Spc}_* \to (\mathsf{Spc}_k)_* \to \mathsf{Sp}_k$ for which we don't introduce any particular name.

Example 4.2. By Exercise 3.13, the square

$$\begin{array}{ccc} (\mathbb{G}_\mathrm{m}, 1) & \xrightarrow{t} & (\mathbb{A}^1, 1) \\ \downarrow{\scriptstyle 1/t} & & \downarrow \\ (\mathbb{A}^1, 1) & \longrightarrow & (\mathbb{P}^1, \infty) \end{array}$$

of open immersion is a cocartesian square in $(\mathsf{Spc}_k)_*$ and we deduce that it remains cocartesian in Sp_k. Since $\mathbb{A}^1 \simeq *$ this implies that $\Sigma\mathbb{G}_{\mathrm{m}} \simeq \mathbb{P}^1 \in \mathsf{Sp}_k$. Since the tensor product is compatible with colimits we may also write this as

$$S^1 \wedge \mathbb{G}_{\mathrm{m}} \simeq \mathbb{P}^1$$

and we see that both 'spheres' on the left become invertible in Sp_k. In particular, Sp_k is *stable*.

Remark 4.3. By the universal property of spectra (Remark 2.56) and the previous example, there is a unique colimit preserving symmetric monoidal functor $\mathsf{Sp} \to \mathsf{Sp}_k$ which we don't baptize either.

Convention 4.4. For $a, b \in \mathbb{Z}$ define the *motivic sphere spectrum* in Sp_k

$$\mathbb{S}^{a,b} := \mathbb{S}^{a-2b} \wedge (\mathbb{P}^1)^{\wedge b} \simeq \mathbb{S}^{a-b} \wedge (\mathbb{G}_{\mathrm{m}})^{\wedge b}$$

of *topological degree* a and *weight* b. For any $E \in \mathsf{Sp}_k$ we also let $\Sigma^{a,b}E$ denote $\mathbb{S}^{a,b} \wedge E$.

Another common notation is $\mathbb{1}(b)[a]$ instead of $\mathbb{S}^{a,b}$.

Example 4.5.

- $\mathbb{S}^{0,0} = \mathbb{S}^0 = \mathbb{S}$
- $\mathbb{G}_{\mathrm{m}} = \mathbb{S}^{1,1}$
- $\mathbb{P}^1 = \mathbb{S}^{2,1}$
- $(\mathbb{A}^n - 0) = \mathbb{S}^{2n-1,n}$

Example 4.6. The complex realization satisfies:

$$\mathfrak{R}_{\mathbb{C}}(\mathbb{S}^{a,b}) = \mathbb{S}^a,$$

which justifies calling a the topological degree.

Exercise 4.7. What is the real realization $\mathfrak{R}_{\mathbb{R}}(\mathbb{S}^{a,b})$?

Example 4.8. Recall Remark 3.34. In $\mathrm{K}_0((\mathsf{Sp}_k)^\omega)$ one has $\Sigma A = -A$ because of the bifiber sequence $A \to 0 \to \Sigma A$. We then deduce:

1. $[M^c(\mathrm{Spec}(k))] = [\mathbb{S}] = 1$
2. $[M^c(\mathbb{P}^1)] = [\mathbb{S} \oplus \mathbb{S}^{-2,-1}] = [\mathbb{S}] + [\mathbb{S}^{-2,-1}] = 1 + [\mathbb{S}^{0,-1}]$ (see Example 4.26)
3. $[M^c(\mathbb{A}^1)] = [M^c(\mathbb{P}^1)] - [M^c(\mathrm{Spec}(k))] = [\mathbb{S}^{0,-1}]$

Remark 4.9. It follows from the last example that $[M^c]$ factors through the localization

$$[M^c]\colon \mathrm{K}_0(\mathrm{Var}_k)[[\mathbb{A}^1]^{-1}] \to \mathrm{K}_0((\mathsf{Sp}_k)^\omega). \tag{4.10}$$

The map is presumably not surjective (example?) although this becomes true if one replaces $(\mathsf{Sp}_k)^\omega$ by the stable subcategory $(\mathsf{Sp}_k)^{\mathrm{fin}}$ generated by $\mathbb{S}^{0,b} \wedge M^c(X)$, $X \in \mathrm{Sch}_k$, $b \in \mathbb{Z}$, of course, cf. [Rön25]. The inclusion $(\mathsf{Sp}_k)^{\mathrm{fin}} \hookrightarrow (\mathsf{Sp}_k)^\omega$ is the idempotent-completion and the induced morphism on K_0 is injective [Tho97].

Some potential elements in the kernel of (4.10) are discussed in [BM25, § 5].

Convention 4.11. We set $\pi_{a,b}(\mathbb{S}) := \mathrm{Hom}_{\mathsf{Sp}_k}(\mathbb{S}^{a,b}, \mathbb{S})$, the (bigraded) homotopy groups of the motivic sphere spectrum.

Remark 4.12. Computing these homotopy groups is difficult. This won't be surprising given that the same is true for the homotopy groups of the sphere spectrum in topology. We now consider a special case: $\pi_{0,0}(\mathbb{S})$.

4.2 Degree

Example 4.13. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a pointed self-map. Under the complex realization, $\mathfrak{R}_{\mathbb{C}}(\Sigma^\infty(f)) \in \pi_0(\mathbb{S}) \cong \mathbb{Z}$ is given by the Brouwer degree of the self-map $f(\mathbb{C})\colon S^2 \to S^2$. If we write $f = g/h$ where $g, h \in k[t]$ are coprime monic polynomials with $n = \deg(g) > \deg(h)$ then

$$\deg(\Sigma^\infty(f(\mathbb{C}))) = n.$$

Similarly, under the real realization, it is given by the Brouwer degree of the self-map $f(\mathbb{R})\colon S^1 \to S^1$. Using the local-degree formula, pick a regular value y. Then $\deg(f(\mathbb{R})) = \sum_{f(\mathbb{R})(x)=y} \pm 1$, where the sign measures whether $f(\mathbb{R})$ is orientation-preserving or -reversing at x.

Remark 4.14. It turns out that both of the values in the above example are encoded by a classical invariant, called the *Bézout form*, which we now recall [Caz12, Definition 3.4]. Write

$$\frac{g(X)h(Y) - g(Y)h(X)}{X - Y} =: \sum_{1 \le p,q \le n} c_{p,q} X^{p-1} Y^{q-1}.$$

(One can show that the left-hand side is indeed a polynomial.) The coefficients $c_{p,q}$ are the entries of a symmetric $(n \times n)$-matrix, corresponding to a non-degenerate symmetric bilinear form $B(f)$.

Recall also that the *Grothendieck-Witt ring* $\mathrm{GW}(k)$ is the group completion of isomorphism classes of non-degenerate symmetric bilinear forms under orthogonal sum. The multiplication is induced by the tensor product of forms. We have therefore produced a map

$$\begin{aligned} \mathrm{End}_*(\mathbb{P}^1) &\to \mathrm{GW}(k), \\ f &\mapsto B(f). \end{aligned} \tag{4.15}$$

Note (?) that the rank of $B(f)$ is $n = \deg(f(\mathbb{C}))$ while the signature of $B(f)$ is $\deg(f(\mathbb{R}))$.

Theorem 4.16 (Morel [Mor12], Cazanave [Caz12])**.** *The map* (4.15) *factors through an isomorphism of rings*

$$\deg^{\mathbb{A}^1}\colon \pi_{0,0}(\mathbb{S}) \xrightarrow{\sim} \mathrm{GW}(k).$$

Remark 4.17.

1. Reflect for a minute on the fact that at no point in defining Sp_k did symmetric bilinear forms occur. It's quite a remarkable isomorphism!
2. Morel actually computes the entire $\mathbb{Z}$-graded ring $\oplus_n \pi_{n,n}(\mathbb{S})$ as the Milnor-Witt K-theory of k. Cazanave gave the isomorphism in degree 0 the explicit form above in terms of the Bézout form.

3. While the theorem shows that

$$\pi_0 \operatorname{Map}_{(\mathsf{Spc}_k)_*}(\mathbb{P}^1, \mathbb{P}^1) \to \pi_0 \operatorname{Map}_{\mathsf{Sp}_k}(\mathbb{S}, \mathbb{S}) = \pi_{0,0}(\mathbb{S})$$

is surjective, it is not injective. The relations come from $(\mathbb{P}^1)^{\wedge 2}$. (In fact, all further $\mathbb{P}^1$-suspensions induce isomorphisms.)

Remark 4.18. There is a canonical map $k^\times \to \mathrm{GW}(k)$ that sends u to the form $(x, y) \mapsto uxy$ denoted $\langle u \rangle$, and these generate the ring, with three relations:

1. $\langle u \rangle \langle v \rangle = \langle uv \rangle$
2. $\langle u^2 v \rangle = \langle v \rangle$
3. $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u + v) \rangle$

whenever these make sense.

Remark 4.19. The fact that these are generators is the statement that every form is (stably) diagonalizable. We explain why this lifts to Sp_k.

Suppose first that $\mathrm{char}(k) \neq 2$. Every symmetric matrix is similar to a diagonal one via a special linear matrix. The latter can be written as a product of elementary matrices each of which is $\mathbb{A}^1$-homotopic to the identity matrix.

If $\mathrm{char}(k) = 2$ the same argument shows that the symmetric matrix is homotopic to a block diagonal matrix with blocks of size at most 2 of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. But the latter is homotopic to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is diagonalizable.

Remark 4.20. Note that $\langle u \rangle$ is the degree of the pointed self-map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^1 \\ (x : y) &\mapsto (x : uy) \end{aligned}$$

where the base point is $\infty = (1 : 0)$.

Example 4.21. If k is algebraically closed then $\mathrm{rank}\colon \mathrm{GW}(k) \xrightarrow{\sim} \mathbb{Z}$. In fact, this is true already when k is quadratically closed.

Example 4.22. For $k = \mathbb{R}$ one has $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \mathbb{Z}/2$ so $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}[X]/(X^2 - 1)$ with $X = \langle -1 \rangle$. The identification is given by rank and signature (that is, the trace of the symmetric matrix). (Every real symmetric form is equivalent to one with ± 1 and 0 on the diagonal only. So the signature is the number of 1's minus the number of -1's. Compare Example 4.13.)

4.3 Enumerative geometry

The $\mathbb{A}^1$-degree feeds into a program of arithmetically enriching results in enumerative geometry. For an introduction see e.g. [Lev22].

Example 4.23. We mention as an example a beautiful work of Kass–Wickelgren [KW21]. Let $X \subseteq \mathbb{P}^3_k$ be a smooth cubic surface. It is a classical result that the number of lines on X is always 27 when $k = \mathbb{C}$. When $k = \mathbb{R}$ the number is not independent of the surface[8] but it was observed not too long ago that a signed count (depending on whether the line is 'hyperbolic' or 'elliptic') *is*. In [KW21], the following identity is established for arbitrary k with $\mathrm{char}(k) \neq 2$:

$$\sum_{u \in L^\times/(L^\times)^2} (\#\text{lines of type } u) \cdot \mathrm{Tr}_{L/k}(\langle u \rangle) = 15\langle 1 \rangle + 12\langle -1 \rangle \in \mathrm{GW}(k)$$

By taking the rank $\mathrm{GW}(\mathbb{C}) \to \mathbb{Z}$ one obtains the classical count of 27. By taking the signature $\mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$ one obtains the signed count of 3 (= 3 − 0 = 5 − 2 = 9 − 6 = 15 − 12). By taking the discriminant $\mathrm{GW}(\mathbb{F}_q) \to \mathbb{Z}/2$ one obtains a parity constraint over a finite field of odd characteristic.

Remark 4.24. Let $(C, \otimes, \mathbb{1})$ be a symmetric monoidal category. An object $c \in C$ is *(strongly) dualizable* if there exists $c^\vee \in C$ and morphisms $\mathbb{1} \to c \otimes c^\vee$, $c^\vee \otimes c \to \mathbb{1}$ such that the composites

$$c \to c \otimes c^\vee \otimes c \to c, \qquad c^\vee \to c^\vee \otimes c \otimes c^\vee \to c^\vee$$

are both the identities. (The dual and the 'unit' and 'counit' are unique up to unique isomorphism if they exist.)

Exercise 4.25. Show that a k-vector space V is dualizable (with respect to the usual tensor product of k-vector spaces) iff it is finite-dimensional.

Example 4.26. The six-functor formalism for Sp_{-} (Section 3.3) implies that $M(X) = X_+ \in \mathsf{Sp}_k$ is dualizable whenever X/k is smooth and proper, with dual $M^c(X)$, see [Rio05].

Remark 4.27. Given an endomorphism $f\colon c \to c$ of a dualizable $c \in C$, one defines its *trace*

$$\mathrm{tr}(f)\colon \mathbb{1} \to c \otimes c^\vee \xrightarrow{f \otimes c^\vee} c \otimes c^\vee \simeq c^\vee \otimes c \to \mathbb{1} \qquad \in \mathrm{End}(\mathbb{1}).$$

The *Euler characteristic* $\chi(c)$ is the trace of the identity on c.

Exercise 4.28. If you did the previous exercise you will have no difficulty verifying that these notions recover the trace of a linear map and the dimension of a (finite-dimensional) vector space, respectively.

Remark 4.29. Using the degree we observe that traces in Sp_k take values in $\mathrm{GW}(k)$ and therefore may encode interesting arithmetic information. This was explored by Hoyois in [Hoy14] to whom the following examples are due.

Example 4.30. Let $f\colon X \to X$ be an endomorphism with X smooth and proper, and assume that the fixed points X^f are étale. Then

$$\mathrm{tr}(f\colon X_+ \to X_+) \;=\; \sum_{x \in X^f} \mathrm{Tr}_{k(x)/k}\langle \det(\mathrm{id} - df_x) \rangle.$$

[8] It can be 3, 7, 15, or 27.

In particular, if $X = \mathrm{Spec}(L)$ with L/k a finite separable extension then

$$\chi(X_+) = \mathrm{Tr}_{L/k}\langle 1\rangle.$$

Example 4.31. If $f\colon X \to X$ is an endomorphism with X smooth and proper, and if $\mathrm{tr}(f\colon X_+ \to X_+) \neq 0$ then f has a fixed point.

On the other hand, if the sheaf of differentials on X has a non-vanishing global section (e.g. an elliptic curve) then $\chi(X_+) = 0$.

Remark 4.32. By [LR20], the Euler characteristic of a smooth and proper X can also be computed in terms of its Hodge cohomology with coherent duality as bilinear form.

Remark 4.33. Let k be of characteristic zero (or admit resolution of singularities). Then all compact objects in Sp_k are also dualizable [Rio05] and so we may compose the association $X \mapsto M^c(X)$ of Remark 3.34 with the Euler characteristic. By the additivity of Euler characteristics [May01], χ descends to K_0 and we get a ring homomorphism

$$\mathrm{K}_0(\mathrm{Var}_k)[[\mathbb{A}^1]^{-1}] \xrightarrow{[M^c]} \mathrm{K}_0((\mathsf{Sp}_k)^{\omega}) \xrightarrow{\chi} \mathrm{GW}(k).$$

This morphism is surjective [BM25, Proposition 7.5].

5 Cohomology theories

5.1 Classical

Convention 5.1. Let $\mathbb{E} \in \mathsf{Sp}_k$. We define $\mathbb{E}$-*cohomology* and $\mathbb{E}$-*homology* of $p\colon X \to \mathrm{Spec}(k)$ in Sch_k as

$$\mathbb{E}^{a,b}(X) := \mathrm{Hom}_{\mathsf{Sp}_k}(\mathbb{S}^{-a,-b}, p_*p^*\mathbb{E}), \qquad \mathbb{E}_{a,b}(X) := \mathrm{Hom}_{\mathsf{Sp}_k}(\mathbb{S}^{a,b}, p_!p^!\mathbb{E}).$$

For $\mathbb{E} = \mathbb{S}$ we also denote $\mathbb{E}_{a,b}(X)$ by $\pi_{a,b}(X)$ and call them *stable motivic homotopy groups*. (This generalizes the bigraded homotopy groups of the motivic sphere spectrum defined earlier which is the case $X = \mathrm{Spec}(k)$.)

Note that if $\mathbb{E} \in \mathrm{CAlg}(\mathsf{Sp}_k)$ (or just a monoid in $\mathrm{ho}(\mathsf{Sp}_k)$) then the cohomology $\mathbb{E}^{*,*}(X)$ becomes a bigraded ring.

Example 5.2. Recall the complex realization $\mathfrak{R}_{\mathbb{C}}\colon \mathsf{Sp}_{\mathbb{C}} \to \mathsf{Sp}$ which has a right adjoint $\mathfrak{R}_{\mathbb{C},*}$. In particular, for any topological cohomology theory, that is, any spectrum $E \in \mathsf{Sp}$, we get an induced cohomology theory $\mathbb{E} := \mathfrak{R}_{\mathbb{C},*}E \in \mathsf{Sp}_{\mathbb{C}}$. For any complex X one then has (non-obviously):

$$\mathbb{E}^{a,b}(X) = E^{a,b}(X(\mathbb{C})).$$

Example 5.3. Take $E = \mathrm{H}A$ or $E = KU \in \mathsf{Sp}$ as in Examples 2.60 and 2.62. It follows that $\mathfrak{R}_{\mathbb{C},*}\mathrm{H}A \in \mathsf{Sp}_{\mathbb{C}}$ represents Betti cohomology with coefficients in A. Similarly, $\mathfrak{R}_{\mathbb{C},*}KU \in \mathsf{Sp}_{\mathbb{C}}$ represents complex topological K-theory.

Remark 5.4. All the examples of cohomology theories mentioned so far were obtained from the complex realization. In fact, there is a close relation between "Weil" cohomology theories and realizations, see [Ayo]. The following examples are conveniently collected in *loc. cit.*

Example 5.5. For any prime ℓ different from the characteristic of k there is a motivic ring spectrum $\mathbb{E}_\ell$ representing ℓ-adic cohomology.

Example 5.6. Let $\mathrm{char}(k) = 0$. There is a motivic ring spectrum $\mathbb{E}_{\mathrm{dR}}$ representing algebraic de Rham cohomology.

Example 5.7. Under suitable assumptions there is a motivic ring spectrum $\mathbb{E}_{\mathrm{rig}}$ representing Berthelot's rigid cohomology.

5.2 Algebraic K-theory

We now turn to MV-cohomology theories that are not as 'classical'. One way to import cohomology theories from topology into algebraic geometry was presented using a topological realization in Section 5.1. However, many of these topological theories afford genuine algebro-geometric analogues. For example, if X is a scheme, there is an *algebraic K-theory* space (or spectrum) $K(X)$ and if $X/\mathbb{C}$ there is a map induced by analytification

$$K(X) \to KU(X(\mathbb{C})),$$

which, however, is not an equivalence in general. In this section we discuss this algebraic K-theory as a MV-cohomology theory, following [Bac21].

Convention 5.8. Let $M \in \mathrm{CAlg}(\mathsf{Spc})$. The functor $\pi_0\colon \mathsf{Spc} \to \mathsf{Set}$ is symmetric monoidal so that $\pi_0(M)$ is a monoid. We say that M is *grouplike* if $\pi_0(M)$ is a group.

By the adjoint functor theorem, the inclusion of grouplike objects $\mathrm{CAlg}_{\mathrm{gp}}(\mathsf{Spc}) \hookrightarrow \mathrm{CAlg}(\mathsf{Spc})$ has a left adjoint $(-)^+$ called *group completion.*

Definition 5.9. Let X/k be a scheme. Vector bundles on X form a symmetric monoidal category under direct sum. Passing to the core (Convention 2.12) we obtain a commutative algebra in Spc that we may group-complete:

$$K^{\oplus}(X) := ((\mathrm{Vect}_X, \oplus)^{\simeq})^+ \in \mathrm{CAlg}_{\mathrm{gp}}(\mathsf{Spc}),$$

which is the direct-sum K-theory space of X. It is canonically pointed at 0.

Example 5.10. We may describe $K_0^{\oplus}(X)$ as the Grothendieck group of the monoid $\pi_0((\mathrm{Vect}_X, \oplus)^{\simeq})$. That is, the free abelian group on isomorphism classes of vector bundles on X modulo the relation $[M] + [N] = [M \oplus N]$.

For example, $K_0(\mathrm{Spec}(k)) = \mathbb{Z}$.

Definition 5.11. Let $K = L_{\mathrm{Zar}} K^{\oplus}\colon (\mathrm{Sm}_k)^{\mathrm{op}} \to \mathsf{Spc}$ be the Zariski sheafification. $K(X)$ is called the *algebraic K-theory space* of $X \in \mathrm{Sm}_k$. (One can also sheafify the functor $(\mathrm{Sm}_k)^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{gp}}(\mathsf{Spc})$ instead. This shows that K has a canonical grouplike infinite loop space structure.)

Remark 5.12. This is arguably not the most enlightening definition of K-theory. But one can show (e.g. [Wei13]):

- For $X = \mathrm{Spec}(R)$ a smooth affine k-scheme, $K(X) \simeq K^{\oplus}(X)$ is the group completion of finitely generated projective R-modules and isomorphisms.
- For general schemes X (not necessarily affine nor regular), the correct (meaning, the accepted) definition of $K(X)$ was given by Thomason–Trobaugh [TT90]. It is an invariant of the stable ∞-category of perfect complexes on X. In any case, it turns out that this restricts to $K\colon (\mathrm{Sm}_k)^{\mathrm{op}} \to \mathsf{Spc}$ defined above. $K_0(X)$ is the free abelian group on isomorphism classes of vector bundles on X modulo the relation $[V_1] + [V_3] = [V_2]$ whenever there is a short exact sequence $V_1 \to V_2 \to V_3$.

Construction 5.13. Let $\alpha \in \mathrm{Vect}_{\mathbb{P}^1}$. For any (smooth) X, tensoring with α yields an additive functor

$$\alpha \otimes -\colon \mathrm{Vect}_X \to \mathrm{Vect}_{\mathbb{P}^1 \times X}$$

and therefore a morphism of commutative monoids in Spc,

$$\alpha\colon K^{\oplus}(X) \to K^{\oplus}(\mathbb{P}^1 \times X),$$

and, Zariski sheafifying,

$$\alpha\colon K \to K(\mathbb{P}^1 \times -).$$

Example 5.14. The tautological line bundle (corresponding to the invertible sheaf $\mathcal{O}(-1)$) defines an object $\gamma \in \mathrm{Vect}_{\mathbb{P}^1}$. Similarly, the trivial line bundle (corresponding to the invertible sheaf $\mathcal{O}$) defines $1 \in \mathrm{Vect}_{\mathbb{P}^1}$. Using the grouplike monoid structure on K we may consider the difference

$$\gamma - 1\colon K \to K(\mathbb{P}^1 \times -)$$

Lemma 5.15. *The map $\gamma - 1$ induces an equivalence*

$$\gamma - 1\colon K \xrightarrow{\sim} \Omega_{\mathbb{P}^1} K \tag{5.16}$$

in $(\mathsf{Spc}_k)_*$.

Proof. (Since $\gamma - 1$ is a morphism of commutative monoids and the basepoint is given by 0, it is clear that $\gamma - 1$ also preserves the basepoint.) Restricting γ to $\infty \in \mathbb{P}^1$ yields the trivial 1-dimensional vector space so that $\gamma|_\infty \simeq 1|_\infty$ via some fixed isomorphism. This yields a homotopy between the composite

$$K \xrightarrow{\gamma-1} K(\mathbb{P}^1 \times -) \xrightarrow{|_\infty} K$$

and 0. By Exercise 3.22, this provides the required lift

$$K \to \Omega_{\mathbb{P}^1} K$$

in $(\mathsf{Spc}_k)_*$. That this is an equivalence is a consequence of the projective bundle theorem for K-theory. □

Definition 5.17. The *algebraic K-theory spectrum* is $\mathrm{KGL} = (K, K, K, \dots) \in \mathsf{Sp}_k$ with bonding equivalences (5.16).

Proposition 5.18. *Let $X \in \mathrm{Sm}_k$. Then* $\mathrm{KGL}^{a,b}(X) = \begin{cases} K_{2b-a}(X) & : 2b - a \geq 0 \\ 0 & : \textit{else} \end{cases}$.

Proof. Note that $\mathbb{P}^1 \wedge (E_0, E_1, \dots) = (E_1, E_2, \dots)$ in Sp_k hence $\mathbb{P}^1 \wedge \mathrm{KGL} = \mathrm{KGL}$ and it follows that

$$\mathbb{S}^{a,b} \wedge \mathrm{KGL} = \mathbb{S}^{a-2b,0} \wedge \mathrm{KGL} = \mathbb{S}^{2b-a,2b-a} \wedge \mathrm{KGL}.$$

If $-p := a - 2b \leq 0$ then

$$\begin{aligned} \mathrm{KGL}^{a,b}(X) &= \mathrm{Hom}_{\mathsf{Sp}_k}(X_+ \wedge \mathbb{S}^{p,0}, \mathrm{KGL}) \\ &= \mathrm{Hom}_{(\mathsf{Spc}_k)_*}(X_+ \wedge \mathbb{S}^{p}, K) \\ &= K_{2b-a}(X). \end{aligned}$$

If $p := a - 2b > 0$ then

$$\begin{aligned} \mathrm{KGL}^{a,b}(X) &= \mathrm{Hom}_{\mathsf{Sp}_k}(X_+ \wedge \mathbb{S}^{p,p}, \mathrm{KGL}) \\ &= \mathrm{Hom}_{(\mathsf{Spc}_k)_*}(X_+ \wedge (\mathbb{G}_m)^{\wedge p}, K). \end{aligned}$$

We will show that this vanishes by induction on $p > 0$. If $p = 1$ we apply $\mathrm{Map}_{(\mathsf{Spc}_k)_*}(X_+ \wedge -, K)$ to the cofiber sequence

$$S^0 \to (\mathbb{G}_m)_+ \to \mathbb{G}_m$$

to get a long exact sequence ending in

$$K_1(X \times \mathbb{G}_m) \to K_1(X) \to \mathrm{KGL}^{a,b}(X) \to K_0(X \times \mathbb{G}_m) \to K_0(X).$$

The last map is an isomorphism by the fundamental theorem of algebraic K-theory. The first map is split surjective by the projection map $X \times \mathbb{G}_m \to X$. It follows that the third term vanishes, as required.

The induction step is similar and we leave it as an exercise. □

Remark 5.19. If X/k is not smooth then $\mathrm{KGL}^{a,b}(X)$ does not compute the correct K-theory groups (Remark 5.12) in general. To see this, observe that the latter are not $\mathbb{A}^1$-invariant for general schemes.

Remark 5.20. There are more geometric models of KGL, in terms of the classifying space of GL_∞ or in terms of Grassmannians. Such models allow to show, among other things, that if $k = \mathbb{C}$ then $\mathfrak{R}_{\mathbb{C}}\mathrm{KGL} = \mathrm{KU}$.

5.3 Motivic cohomology

We can also ask for an algebro-geometric analogue of singular cohomology. To find the correct analogue let us see which descriptions of $\mathrm{H}\mathbb{Z} \in \mathsf{Sp}$ can be translated.

Remark 5.21.

1. Given that $\mathrm{H}\mathbb{Z} = (K(\mathbb{Z}, n))_{n\geq 0}$ we could try to find analogues of the Eilenberg–MacLane spaces in $(\mathsf{Spc}_k)_*$ together with bonding equivalences.
2. In Exercise 2.61 you proved that $\pi_*(\mathrm{H}\mathbb{Z})$ is concentrated in degree 0 and equal to $\mathbb{Z}$. This singles out this spectrum as an object of the heart of the homotopy t-structure on Sp. That is, let $\mathsf{Sp}_{\geq n} \subseteq \mathsf{Sp}$ denote the subcategory generated by $\mathbb{S}^n$ under colimits and extensions. By the adjoint functor theorem, the inclusion $\mathsf{Sp}_{\geq n} \hookrightarrow \mathsf{Sp}$ admits a right adjoint and we denote the composite $f_n \colon \mathsf{Sp} \to \mathsf{Sp}_{\geq n} \hookrightarrow \mathsf{Sp}$. Clearly, we get a filtration
$$\cdots \to f_{n+1} \to f_n \to \cdots \tag{5.22}$$
indexed by the integers, and we denote by $s_n := \mathrm{cof}(f_{n+1} \to f_n)$ the *nth slice*. It turns out that s_0 factors as
$$s_0 \colon \mathsf{Sp} \xrightarrow{\pi_0} \mathrm{Mod}_{\mathbb{Z}} \xrightarrow{\mathrm{H}} \mathsf{Sp}.$$
It follows that $s_0(\mathbb{S}) = \mathrm{H}\mathbb{Z}$. This suggests an approach would be to find a filtration on Sp_k analogous to (5.22).
3. Singular homology associates to any topological space $\mathcal{X}$ a chain complex $C_*(\mathcal{X}) \in \mathsf{D}(\mathbb{Z})$ in the derived category of abelian groups. This descends to an adjunction
$$L \colon \mathsf{Sp} \rightleftarrows \mathsf{D}(\mathbb{Z}) : R$$
and $\mathrm{H}\mathbb{Z} = RL(\mathbb{S})$. This suggests looking for an analogue of $\mathsf{D}(\mathbb{Z})$ in algebraic geometry.
4. The complex $C_*(\mathcal{X})$ may be expressed in terms of the singular simplicial set $X = \mathrm{Hom}(|\Delta^\bullet|, \mathcal{X})$ of Alternative 2.1. This suggests looking for an algebraic version of the simplices $|\Delta^n|$ in order to define an analogue of singular homology.

Remark 5.23. Each of these approaches can be implemented:

1. Voevodsky [Voe98; Voe10] defined Eilenberg–MacLane spaces $K(\mathbb{Z}(n), 2n) \in (\mathsf{Spc}_k)_*$ using an analogue of symmetric powers in algebraic geometry, inspired by the Dold–Thom theorem in topology. The bonding maps $\mathbb{P}^1 \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(n+1), 2n+2)$ can be defined quite easily but showing that their adjoints are equivalences is hard (and requires resolution of singularities). We may consider the associated spectrum $\mathrm{H}\mathbb{Z}^{\mathrm{em}} \in \mathsf{Sp}_k$.
2. There are different filtrations on Sp_k one might try but the correct one for this purpose is by letting $(\mathsf{Sp}_k)_{\geq n}$ denote the stable subcategory generated by $X_+ \wedge (\mathbb{P}^1)^{\wedge n}$ for $X \in \mathrm{Sm}_k$ under colimits. This is called the *slice filtration* [Voe02]. One can then define $\mathrm{H}\mathbb{Z}^{\mathrm{sf}} := s_0(\mathbb{S}) \in \mathsf{Sp}_k$ [Voe04; Lev08].
3. Hanamura [Han95], Levine [Lev98] and Voevodsky [Voe00] define[9] the *derived* ∞*-category of mixed motives* DM_k and there is a functor $L \colon \mathsf{Sp}_k \to \mathsf{DM}_k$ that 'adds transfers (=multi-valued maps) and linearizes'. We may define $\mathrm{H}\mathbb{Z}^{\mathrm{dm}} = RL(\mathbb{S})$ where R denotes the right adjoint to L [Hoy15].

[9] Two caveats: Not all of them originally came with an ∞-categorical enhancement, and they are only (anti-)equivalent. To be specific, we use Voevodsky's.

4. Suslin and Voevodsky [SV96] use the algebraic simplices $\Delta^n_{\mathrm{alg}} = \{x_0 + \cdots + x_n = 1\} \subseteq \mathbb{A}^{n+1}$ and 'multivalued maps' $\Delta^n_{\mathrm{alg}} \to X$ to define the *Suslin homology* of X.

Somewhat miraculously and somewhat by design they all give rise to essentially the same theory, namely *motivic (co)homology*. To be more precise the above references show:

Theorem 5.24. *Assume k of characteristic zero or invert the characteristic of the field. Then:*

$$\mathrm{H}\mathbb{Z}^{em} \simeq \mathrm{H}\mathbb{Z}^{ss} \simeq \mathrm{H}\mathbb{Z}^{dm} \tag{5.25}$$

Definition 5.26. We denote by $\mathrm{H}\mathbb{Z} \in \mathsf{Sp}_k$ this common spectrum, called the *motivic cohomology spectrum*.

Remark 5.27. One may endow $\mathrm{H}\mathbb{Z}$ with the structure of a commutative algebra. This can be done in all three approaches above canonically and they all coincide. In particular, one may consider modules over this algebra, and [RØ08; HKØ17] show (under the same assumptions as above) that

$$\mathrm{Mod}_{\mathrm{H}\mathbb{Z}} \simeq \mathsf{DM}_k \,.$$

This gives an approach to the derived ∞-category of mixed motives without mentioning transfers (although the comparisons (5.25) aren't as puristic). Of course, this is also analogous to the equivalence $\mathrm{Mod}_{\mathrm{H}\mathbb{Z}} \simeq \mathsf{D}(\mathbb{Z})$ in topology.

Example 5.28. Let X be a smooth k-scheme. Then

$$\mathrm{H}\mathbb{Z}^{2n,n}(X) \simeq \mathrm{CH}^n(X)$$

identifies with the codimension-n Chow groups of X.

Remark 5.29. This leads to the observation that Chow motives embed fully faithfully into the derived category of mixed motives. By the former we understand the additive ordinary category CHM_k which has objects smooth projective k-schemes, and morphisms $\mathrm{Hom}(X, Y) = \mathrm{CH}_d(X \times Y)$ where X is of pure dimension d. There is a functor

$$\mathrm{CHM}_k \hookrightarrow \mathrm{ho}(\mathsf{DM}_k), \qquad X \mapsto \mathrm{H}\mathbb{Z} \wedge X_+$$

which is fully faithful. This creates the bridge between Grothendieck's pure motives (which are conjecturally an additive quotient of CHM_k) and the theory described in these notes.

Remark 5.30. Let us also mention the relation with the last approach to motivic (co)homology. For any smooth X/k, its Suslin homology, it turns out, is computed as

$$\mathrm{H}^{\mathrm{sing}}_n(X) \simeq \mathrm{H}\mathbb{Z}_{n,0}(X).$$

Remark 5.31. Let $k = \mathbb{C}$. Then $\mathfrak{R}_{\mathbb{C}}\mathrm{H}\mathbb{Z} = \mathrm{H}\mathbb{Z}$.

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