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The Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara

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Introduction

Let k be an algebraically closed field and let us work in the category $\mathbf{Sch}/k$ of separated finite type k-schemes. Let us be given a correspondence $c = (c_1, c_2) : C \to X \times X$, a complex of l-adic sheaves $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X, \mathbb{Q}_l)$, where l is invertible in k, and a cohomological correspondence $u : \mathbf{R}c_{2!}c_1^*\mathscr{F} \to \mathscr{F}$. Assume that both X and c_1 are proper.

The correspondence c gives rise to the scheme of fixed points $\operatorname{Fix}(c) \subset C$ which is defined as $c^{-1}(\operatorname{diag} X)$. Also, u gives rise to an endomorphism $\mathbf{R}\Gamma_c(u) : \mathbf{R}\Gamma_c(X, \mathscr{F}) \to \mathbf{R}\Gamma_c(X, \mathscr{F})$ of the cohomology with compact support on X with values in $\mathscr{F}$. Representing $\mathbf{R}\Gamma_c(X, \mathscr{F})$ by a projective system of perfect complexes of $\mathbb{Z}/l^\nu\mathbb{Z}$-modules ($\nu \geq 1$), we may assign to $\mathbf{R}\Gamma_c(u)$ its trace $\operatorname{Tr}(\mathbf{R}\Gamma_c(u)) \in \mathbb{Q}_l$. In this setting, the Lefschetz-Verdier trace formula [14, III, 4.7–8] expresses this global term as a sum of "local terms" on $\operatorname{Fix}(c)$:

$$\operatorname{Tr}(\mathbf{R}\Gamma_c(u)) = \sum_{\beta \in \pi_0(\operatorname{Fix}(c))} \operatorname{lt}_\beta(u).$$

Unfortunately, these local terms are not easy to compute for general correspondences. At the same time this result fails if X or c_1 is not proper. Let us sketch possible strategies (one in each case) to cope with these "defects":

- One obvious way to accommodate a non-proper scheme X is to compactify and "extend" $\mathscr{F}$ and u "by zero". One would hope that neither the global trace nor the local terms change under this operation.

- Suppose now that c_1, although not necessarily proper on the whole of C, admits an open subset $U \subset X$ such that $c_1|_{c_1^{-1}(U)} : c_1^{-1}(U) \to U$ is proper and $\mathscr{F}|_{X \setminus U} = 0$. In this case the two objects $\mathbf{R}\Gamma_c(X, \mathscr{F})$ and $\mathbf{R}\Gamma_c(U, \mathscr{F}|_U)$ are isomorphic and u still gives rise to an endomorphism $\mathbf{R}\Gamma_c(u)$ of $\mathbf{R}\Gamma_c(X, \mathscr{F})$. "Restricting" c and u to $c_1^{-1}(U) \subset C$, one would again hope that the local terms remain the same.

- As for the non-explicitness of the local terms, here is a situation where one has a natural guess for how to compute them. Suppose that y is an isolated fixed point of c, and that c_2 is quasi-finite. In this case, u gives rise to an endomorphism u_y of the stalk $\mathscr{F}_{\overline{c_2 y}}$. Its trace is called the "naive local term of u at y". One's task is then to find conditions which guarantee that the fixed points are isolated and that the naive and the true local terms coincide.

In [19], Varshavsky stated a set of conditions and proved them sufficient to carry out the strategies just sketched, i. e. he proved that this set of conditions suffices to express the global trace associated to $\mathbf{R}\Gamma_c(u)$ as a finite sum of naive local terms, even for X and c_1 not necessarily proper. A bit more explicitly (we won't give *all* the conditions here in the introduction), he assumes k to be the algebraic closure of a finite field, and he also assumes that c is defined over the finite field. Thus, one has at one's disposal the geometric Frobenius which one may use to "twist" c_1. Deligne conjectured that twisting c_1 by a sufficiently high power n of the Frobenius, and assuming c_2 quasi-finite and c_1 proper, will ensure that the global term is a finite sum of naive local terms. The main result of [19] generalizes this by allowing c_1 to be non-proper as explained above. (The condition not mentioned here explicitly is related to this generalization.) As to the number n of necessary "twists", the article gives a relatively explicit upper bound on it, depending on the correspondence only. Weaker results had been obtained before by Pink in [18] and Fujiwara in [9], and a significant part of the strategy employed in [19] is already present in these two articles. The reader is referred to the introductions of all these articles for some information on the role played by the Lefschetz-Verdier trace formula and the calculation of local terms in algebraic geometry.

In the pages to come, I give a detailed account of [19]. Mathematically, there is (at least from an expert's point of view) nothing new here in comparison with [19]. I have reorganized the exposition at some points, silently corrected a few (minor) mistakes (and most certainly introduced others), and mainly given much more details.

I would like to thank Andrew Kresch and Joseph Ayoub for their assistance during the time I have been working on my master's thesis.

Let me end the introduction by giving an outline of the rest of the document. After some preliminaries in section 1, correspondences and cohomological correspondences are introduced in section 2 (§1, §2). As seen above, under the first two bullets, we will need operations of some kind of proper pushforward and pullback on cohomological correspondences. Apart from these (§3, §4), the second section contains a discussion of the specialization operation on cohomological correspondences (§5) and its application to the deformation to the normal cone construction (§6). Specialization to the normal cone is a main ingredient in proving the vanishing of some local terms. The last paragraph §7 of the second section proves a result of Verdier [20] which says that specialization to the normal cone commutes with restriction to the zero section.

The third section gives the definition of a general "trace map" from which the local terms are obtained by "integration", and which also yields the global trace term when applied to $\mathbf{R}\Gamma_c(u)$ (§1). It is of course important that this trace map behaves well under the operations on cohomological correspondences discussed in the second section. It is in fact proved that it commutes with something called a "cohomological morphism", which subsumes all these operations (§2–§5). Moreover, the Lefschetz-Verdier trace formula is seen to follow easily from the naturality of the trace map with respect to proper pushforward. Finally, the additivity of the trace map is deduced from the additivity of its filtered counterpart, which is in turn proved in §6.

The last section starts by introducing the key concept of a "contracting correspondence" and by relating it to the construction of the deformation to the normal cone (§1, §2). Roughly, for correspondences which are contracting in a neighborhood of their fixed points, the local terms equal the naive local terms. In §3, it is shown that correspondences over finite fields can be made contracting by twisting them with a sufficiently high power of the Frobenius, while §4 concludes this document with the proof of the generalization of Deligne's conjecture explained above.

Contents

1 Preliminaries

In this first section we will fix our notation and recall some concepts and facts used later on. Except for §1 and §2, the reader is advised to skip this section and come back to it if and when needed. We will facilitate this by referring to the individual paragraphs of this section at the relevant places in the main body of the text. (See also the list of notations at the end of the document.)

§1 Schemes

(a) Most of the time, and if not explicitly mentioned otherwise, all schemes considered in the sequel are separated and of finite type over a fixed algebraically closed field k. Seemingly absolute notions from scheme theory are always to be understood relative to k, e. g. morphisms of schemes are morphisms of schemes over k, direct products of schemes are fiber products of schemes over k. Another way to say this is that all statements about schemes and their morphisms are to be understood as applying to the category of separated schemes of finite type over k, denoted $\mathbf{Sch}/k$ (except that we will be sloppy and identify morphisms $X \to k$ with their source X, as usual).

Sometimes, we also work in the category of separated schemes of finite type over a fixed discrete valuation ring R (satisfying some properties yet to be stated), accordingly denoted $\mathbf{Sch}/R$. More explicitly, this is the case in 1.§4, 2.§5–§7 and at a few places in section 3. However, we will always make clear when we change the category.

(b) The scheme defined by k (i. e. the terminal object in $\mathbf{Sch}/k$) is again denoted k. For any scheme X (separated and of finite type over k) we denote by X_{red} the associated reduced scheme and by $\pi_X : X \to k$ its structure morphism. Similar conventions apply to $\mathbf{Sch}/R$. The diagonal morphism $X \hookrightarrow X \times X$ is denoted Δ_X. For a morphism of schemes $f : X \to Y$ we denote both morphisms of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and $f^*\mathcal{O}_Y \to \mathcal{O}_X$ by $f^\sharp$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules then we denote by $f^\circledast\mathcal{F}$ the inverse image of $\mathcal{F}$ by f, i. e. $f^\circledast\mathcal{F} = f^*\mathcal{F} \otimes_{f^*\mathcal{O}_Y} \mathcal{O}_X$.

If $Z \subset Y$ is a closed subscheme then $\mathcal{I}_Z \subset \mathcal{O}_Y$ denotes the ideal sheaf defining Z. For a morphism of schemes $f : X \to Y$ we denote the inverse image ideal sheaf of $\mathcal{I}_Z$ by $f^*\mathcal{I}_Z \cdot \mathcal{O}_X$. It is defined as the ideal sheaf generated by the image of $f^*\mathcal{I}_Z \hookrightarrow f^*\mathcal{O}_Y \to \mathcal{O}_X$, where the second morphism is $f^\sharp$. The scheme-theoretic inverse image of Z along f is denoted $f^{-1}(Z)$, i. e. $f^{-1}(Z) = X \times_Y Z$. The relationship between $f^{-1}(Z)$ and $f^*\mathcal{I}_Z \cdot \mathcal{O}_X$ is given by $\mathcal{I}_{f^{-1}(Z)} = f^*\mathcal{I}_Z \cdot \mathcal{O}_X$.

§2 Derived category

(c) We fix a prime l invertible in k, and a commutative ring with identity Λ which is finite and annihilated by some power of l. The generalization of our main results to the case where Λ is a finite extension of either $\mathbb{Z}_l$ or $\mathbb{Q}_l$ is immediate. (In fact, most of the proofs in this document go through word for word in this context.)

(d) To any scheme X we may associate the derived category of sheaves of Λ-modules for the étale topology on X, denoted $\mathfrak{D}(X, \Lambda)$, and its full "subcategories of bounded above complexes" $\mathfrak{D}^-(X, \Lambda)$ as well as of "complexes of finite tor-dimension with constructible cohomology", denoted $\mathfrak{D}^b_{ctf}(X, \Lambda)$ (or simply $\mathfrak{D}^b_{ctf}(X)$). Recall that the objects of $\mathfrak{D}^b_{ctf}(X)$ are exactly those complexes $\mathcal{F} \in \mathfrak{D}^-(X, \Lambda)$ which satisfy any of the two following equivalent conditions ([6, Rapport, 4.6]):

1. $\mathcal{F}$ is isomorphic (in $\mathfrak{D}^-(X, \Lambda)$) to a bounded complex of sheaves which are both flat and constructible.

2. $\mathscr{F}$ has finite tor-dimension and the cohomology sheaves $\underline{\mathrm{H}}^i(\mathscr{F})$ are constructible.

In particular, if $X = k$ then $\mathfrak{D}^b_{ctf}(X, \Lambda)$ is equivalent to the category of bounded complexes of sheaves of projective finite type Λ-modules with morphisms taken modulo homotopy, denoted $K_{\mathrm{parf}}(\Lambda)$ in [6, Rapport, 4.3]. From the second description above it is easy to see that $\mathfrak{D}^b_{ctf}(X)$ is a triangulated subcategory of $\mathfrak{D}^-(X, \Lambda)$.

Since we have imposed the right "finiteness conditions" (the schemes are of finite type over a field), the categories $\mathfrak{D}^b_{ctf}(X)$ are stable under the six operations

$$f^*, f_*, f^!, f_!, \otimes, \mathbf{R}\underline{\mathrm{Hom}}$$

(see [7, 1.1.2–3]). Here, of course, we abuse notation by writing the same symbol for the functor as for its derived counterpart. The transitivity isomorphisms for these functors (e. g. $(gf)^* \cong f^*g^*$) will usually not be denoted specifically. Recall also the "cocycle condition" for the transitivity isomorphisms (see e. g. [2, XVII, 5.1.8 (i); XVIII, 3.1.13]).

(e) For a scheme X we denote by $\Lambda_X \in \mathfrak{D}^b_{ctf}(X)$ (or just Λ) the constant sheaf associated to Λ, by $K_X = \pi^!_X(\Lambda_k)$ the dualizing complex of X, and by $\mathbb{D}_X = \mathbf{R}\underline{\mathrm{Hom}}(-, K_X)$ (or simply $\mathbb{D}$) the Verdier dual of X. Recall that there is a natural isomorphism of endofunctors of $\mathfrak{D}^b_{ctf}(X)$, $\mathbb{D}^2 \to \mathbb{1}$ (see [6, Th. finitude, 4.3]). Instead of $\pi_{X!}$ we also write $\mathbf{R}\Gamma_c(X, -)$, instead of $\mathrm{H}^i \circ \pi_{X*}$ we write $\mathrm{H}^i(X, -)$, $i \in \mathbb{Z}$. If $j : U \hookrightarrow X$ is an immersion and $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ we often denote $j^*\mathscr{F}$ by $\mathscr{F}|_U$.

(f) For a morphism of schemes $f : X \to Y$, there are well-known adjointness relations

$$f^* \dashv f_*, \quad f_! \dashv f^!$$

of the functors between the categories $\mathfrak{D}^b_{ctf}(X)$ and $\mathfrak{D}^b_{ctf}(Y)$. We denote the units and counits of these adjunctions ambiguously adj. Moreover,

$$(-) \otimes \mathscr{F} \dashv \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, -)$$

for any $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and we denote by ev the adjunction morphism (counit) $\mathscr{F} \otimes \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) \to \mathscr{G}$ derived from the morphism on the level of complexes

$$\begin{aligned} \mathscr{F} \otimes \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) &\longrightarrow \mathscr{G} \\ x \otimes f &\longmapsto f(x) \end{aligned}$$

(see [14, III, 2.2]).

Recall also that $f_! = f_*$ in the case f is proper ([2, XVII, 5.1.8]), $f^! = f^*$ in the case f is étale ([2, XVIII, 3.1.8 (iii)]) and $f^!$ is the functor "sections with support on X" if f is a closed immersion ([2, XVIII, 3.1.8 (ii)]).

§3 Some morphisms

(g) Let $f : X \to Y$ be a morphism of schemes, $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(X)$, and $\mathscr{F}, \mathscr{G} \in \mathfrak{D}^b_{ctf}(Y)$.

If $\mathscr{M}, \mathscr{N}$ are two étale Λ-sheaves on Y, and if V is étale over X, there is a canonical Λ-bilinear map

$$\varinjlim_U \mathscr{M}(U) \times \varinjlim_U \mathscr{N}(U) \longrightarrow \varinjlim_U (\mathscr{M}(U) \times \mathscr{N}(U)) \longrightarrow \varinjlim_U (\mathscr{M}(U) \otimes_\Lambda \mathscr{N}(U)),$$

where the limit is taken over those U étale over Y through which the morphism $V \to X \to Y$ factors. This map induces a morphism of sheaves

$$f^*\mathscr{M} \otimes_\Lambda f^*\mathscr{N} \longrightarrow f^*(\mathscr{M} \otimes_\Lambda \mathscr{N}),$$

which is in fact an isomorphism. Since f^* is exact and takes flat objects to flat objects, it induces in an obvious way an isomorphism of functors between the derived categories,

$$t_{f^*} : f^*\mathscr{F} \otimes f^*\mathscr{G} \xrightarrow{\cong} f^*(\mathscr{F} \otimes \mathscr{G}). \tag{1.1}$$

We denote by t_{f_*} the morphism obtained by adjunction:

$$f_*\mathscr{A} \otimes f_*\mathscr{B} \xrightarrow{\text{adj}} f_*f^*(f_*\mathscr{A} \otimes f_*\mathscr{B}) \underset{\cong}{\xrightarrow{t_{f^*}^{-1}}} f_*(f^*f_*\mathscr{A} \otimes f^*f_*\mathscr{B}) \xrightarrow{\text{adj}} f_*(\mathscr{A} \otimes \mathscr{B}). \tag{1.2}$$

(h) There is a canonical morphism, called the *projection formula*,

$$\text{proj} : f_*\mathscr{A} \otimes \mathscr{F} \xrightarrow{\text{adj}} f_*f^*(f_*\mathscr{A} \otimes \mathscr{F}) \underset{\cong}{\xrightarrow{t_{f^*}^{-1}}} f_*(f^*f_*\mathscr{A} \otimes f^*\mathscr{F}) \xrightarrow{\text{adj}} f_*(\mathscr{A} \otimes f^*\mathscr{F}). \tag{1.3}$$

It is an isomorphism e. g. if $f = \pi_X$ is the structure morphism of a scheme X (see the proof of [2, XVII, 5.2.11]). Moreover, if f is proper, it coincides with the following isomorphism,

$$\text{proj} : f_!\mathscr{A} \otimes \mathscr{F} \xrightarrow{\cong} f_!(\mathscr{A} \otimes f^*\mathscr{F}) \tag{1.4}$$

([2, XVII, 5.2.9]), also called the *projection formula*.

The composition

$$f_!(f^!\mathscr{F} \otimes f^*\mathscr{G}) \underset{\cong}{\xrightarrow{\text{proj}^{-1}}} f_!f^!\mathscr{F} \otimes \mathscr{G} \xrightarrow{\text{adj}} \mathscr{F} \otimes \mathscr{G}$$

defines by adjunction a morphism

$$t_{f^!} : f^!\mathscr{F} \otimes f^*\mathscr{G} \longrightarrow f^!(\mathscr{F} \otimes \mathscr{G}), \tag{1.5}$$

and the composition

$$f^!\,\mathbf{R}\underline{\text{Hom}}(\mathscr{F},\mathscr{G}) \otimes f^*\mathscr{F} \xrightarrow{t_{f^!}} f^!(\mathbf{R}\underline{\text{Hom}}(\mathscr{F},\mathscr{G}) \otimes \mathscr{F}) \xrightarrow{\text{ev}} f^!\mathscr{G}$$

defines, also by adjunction, a morphism

$$\text{ind} : f^!\,\mathbf{R}\underline{\text{Hom}}(\mathscr{F},\mathscr{G}) \xrightarrow{\cong} \mathbf{R}\underline{\text{Hom}}(f^*\mathscr{F}, f^!\mathscr{G}), \tag{1.6}$$

which is an isomorphism by [2, XVIII, 3.1.12.2], called the *induction isomorphism*.

(i) Let $f : X \to Y$ be a morphism of schemes. There is a canonical isomorphism

$$\varepsilon : f^*\Lambda_Y \xrightarrow{\cong} \Lambda_X$$

defined as follows. Denote by f^p the inverse image functor of presheaves, by Λ_W^p the constant presheaf on the scheme W associated to Λ, and by a the sheafification functor. Then the inverse of ε is

$$\Lambda_X = a\Lambda_X^p = af^p\Lambda_Y^p \xrightarrow{\cong} af^pa\Lambda_Y^p = f^*\Lambda_Y,$$

induced by the canonical morphism of presheaves $\Lambda_Y^p \to a\Lambda_Y^p$. The morphism corresponding to ε by adjointness will be denoted $\varepsilon' : \Lambda_Y \to f_*\Lambda_X$.

If $g : Y \to Z$ is a second morphism of schemes then ε induces the following commutative diagram:

$$\begin{CD} \Lambda_X @<{\varepsilon}<{\cong}< (gf)^*\Lambda_Z \\ @A{\varepsilon}A{\cong}A @A{\cong}AA \\ f^*\Lambda_Y @<{\cong}<{\varepsilon}< f^*g^*\Lambda_Z \end{CD} \tag{1.7}$$

Moreover, if $\mathscr{M}$ is an étale sheaf on X then the following square commutes

$$\begin{CD} f_*\mathscr{M} @<{\cong}<< f_*(\mathscr{M} \otimes \Lambda_X) \\ @A{\cong}AA @AA{t_{f_*}}A \\ f_*\mathscr{M} \otimes \Lambda_Y @>>{\varepsilon'}> f_*\mathscr{M} \otimes f_*\Lambda_X \end{CD} \tag{1.8}$$

It follows by adjointness that if $\mathscr{N}$ is an étale sheaf on Y then

$$\begin{CD} f^*\mathscr{N} @<{\cong}<< f^*(\mathscr{N} \otimes \Lambda_Y) \\ @A{\cong}AA @AA{t_{f^*}}A \\ f^*\mathscr{N} \otimes \Lambda_X @<<{\varepsilon}< f^*\mathscr{N} \otimes f^*\Lambda_Y \end{CD}$$

is also commutative. Since the functor of sheaves f^* is exact and preserves flat objects, the same diagram commutes in the derived context as well. One then deduces again by adjointness that the diagram corresponding to (1.8) in the derived context is also commutative.

(j) If f is a proper morphism, we define the *integration map*

$$\int_f : \mathrm{H}^0(X, K_X) \longrightarrow \mathrm{H}^0(Y, K_Y) \tag{1.9}$$

as the composition

$$\mathrm{H}^0(X, K_X) \xrightarrow{\cong} \mathrm{H}^0(Y, f_! f^! K_Y) \xrightarrow{\mathrm{adj}} \mathrm{H}^0(Y, K_Y).$$

If $f = \pi_X$ is the structure morphism of a scheme X, we also denote $\int_{\pi_X}$ simply by $\int_X$.

(k) Let

$$\begin{CD} X @<{g'}<< X' \\ @V{f}VV @VV{f'}V \\ Y @<<{g}< Y' \end{CD} \tag{1.10}$$

be a commutative diagram of morphisms of schemes. We shall now list the base change morphisms we will need in the following sections. Each one will be denoted bc followed by the equation number referring to the equation where it is defined.

Without any assumptions on the morphisms there are base change morphisms

$$f'_! g'^! \longrightarrow g^! f_!, \tag{1.11}$$
$$g^* f_* \longrightarrow f'_* g'^*, \tag{1.12}$$

the first defined in [2, XVIII, 3.1.13.2], the second defined in [2, XVII, 4.1.5].

If the square (1.10) is cartesian, then there are two isomorphisms

$$g^* f_! \xrightarrow{\cong} f'_! g'^*, \tag{1.13}$$
$$f'_* g'^! \xrightarrow{\cong} g^! f_* \tag{1.14}$$

([2, XVIII, 3.1.14.1 and 3.1.12.3]). From these we obtain by adjunction two other morphisms

$$f'^* g^! \longrightarrow g'^! f^*, \tag{1.15}$$
$$f_! g'_* \longrightarrow g_* f'_! \tag{1.16}$$

([2, XVIII, 3.1.14.2] for the first one). If f is proper then (1.16) is the usual transitivity isomorphism and (1.13) coincides with (1.12). If g is smooth and the square cartesian then both (1.12) and (1.15) are isomorphisms. If g is étale then (1.15) is the usual transitivity morphism, (1.14) is the inverse of (1.12), and (1.13) is the inverse of (1.11).

Note that all these base change morphisms behave well with respect to composition of morphisms in the sense of [2, XII, 4.4] (see e. g. [2, XVII, 5.2.4–5; XVIII.3.1.14]).

(I) For two schemes X_i, $i = 1, 2$, we denote the projections $X_1 \times X_2 \to X_i$ by p_i. For $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$ we set $\mathscr{F}_1 \boxtimes \mathscr{F}_2 := p_1^* \mathscr{F}_1 \otimes p_2^* \mathscr{F}_2$. We will need the following constructions from [14, III].

The composition

$$p_1^* \pi^!_{X_1} \Lambda_k \otimes p_2^* \mathscr{F}_2 \xrightarrow{(1.15)} p_2^! \pi^*_{X_2} \Lambda_k \otimes p_2^* \mathscr{F}_2 \xrightarrow{t_{p_2^!}} p_2^! (\Lambda_{X_2} \otimes \mathscr{F}_2) \xrightarrow{\cong} p_2^! \mathscr{F}_2$$

defines an isomorphism

$$K_{X_1} \boxtimes \mathscr{F}_2 \xrightarrow{\cong} p_2^! \mathscr{F}_2 \tag{1.17}$$

by [14, III, 1.7.4]. This also gives rise to an identification (cf. [14, III, 3.1.1])

$$\mathbb{D}\mathscr{F}_1 \boxtimes \mathscr{F}_2 \xrightarrow{\cong} \mathbf{R}\underline{\mathrm{Hom}}(p_1^* \mathscr{F}_1, p_2^* \mathscr{F}_2), \tag{1.18}$$

defined by adjunction as follows:

$$p_1^* \mathscr{F}_1 \otimes (\mathbb{D}\mathscr{F}_1 \boxtimes \mathscr{F}_2) \xrightarrow{\cong} (p_1^* \mathscr{F}_1 \otimes p_1^* \mathbb{D}\mathscr{F}_1) \otimes p_2^* \mathscr{F}_2 \xrightarrow[\cong]{t_{p_1^*}} p_1^* (\mathscr{F}_1 \otimes \mathbb{D}\mathscr{F}_1) \otimes p_2^* \mathscr{F}_2 \xrightarrow{=}$$
$$(\mathscr{F}_1 \otimes \mathbb{D}\mathscr{F}_1) \boxtimes \mathscr{F}_2 \xrightarrow{\mathrm{ev}} K_{X_1} \boxtimes \mathscr{F}_2 \xrightarrow{(1.17)} p_2^! \mathscr{F}_2.$$

If $f : X \to Y$ and $\mathscr{F}, \mathscr{G} \in \mathfrak{D}^b_{ctf}(Y)$ there is a canonical morphism at the level of complexes $f^* \underline{\mathrm{Hom}}^\cdot(\mathscr{F}, \mathscr{G}) \to \underline{\mathrm{Hom}}^\cdot(f^* \mathscr{F}, f^* \mathscr{G})$ which can be derived to yield

$$f^* \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}(f^* \mathscr{F}, f^* \mathscr{G}). \tag{1.19}$$

Also, if $\mathscr{F}_i, \mathscr{G}_i \in \mathfrak{D}^b_{ctf}(X)$, then the composition

$$\mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1, \mathscr{G}_1) \otimes \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_2, \mathscr{G}_2) \xrightarrow{\cong}$$
$$\mathscr{F}_1 \otimes \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1, \mathscr{G}_1) \otimes \mathscr{F}_2 \otimes \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_2, \mathscr{G}_2) \xrightarrow{\mathrm{ev} \otimes \mathrm{ev}} \mathscr{G}_1 \otimes \mathscr{G}_2$$

gives rise, by adjunction, to a morphism

$$\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1,\mathscr{G}_1)\otimes\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_2,\mathscr{G}_2)\longrightarrow\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1\otimes\mathscr{F}_2,\mathscr{G}_1\otimes\mathscr{G}_2). \tag{1.20}$$

Now, let $\mathscr{F}_i,\mathscr{G}_i\in\mathfrak{D}^b_{ctf}(X_i)$. Composing the two morphisms above yields

$$\begin{aligned}\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1,\mathscr{G}_1)\boxtimes\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_2,\mathscr{G}_2)&\xrightarrow{=}p_1^*\,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1,\mathscr{G}_1)\otimes p_2^*\,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_2,\mathscr{G}_2)\\&\xrightarrow{(1.19)\otimes(1.19)}\mathbf{R}\underline{\mathrm{Hom}}(p_1^*\mathscr{F}_1,p_1^*\mathscr{G}_1)\otimes\mathbf{R}\underline{\mathrm{Hom}}(p_2^*\mathscr{F}_2,p_2^*\mathscr{G}_2)\\&\xrightarrow{(1.20)}\mathbf{R}\underline{\mathrm{Hom}}(p_1^*\mathscr{F}_1\otimes p_2^*\mathscr{F}_2,p_1^*\mathscr{G}_1\otimes p_2^*\mathscr{G}_2)\\&\xrightarrow{=}\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}_1\boxtimes\mathscr{F}_2,\mathscr{G}_1\boxtimes\mathscr{G}_2).\end{aligned} \tag{1.21}$$

By [14, III, 2.3], this is an isomorphism.

§4 Nearby cycle functor

We will here recall the construction of the "nearby cycle functor" (cf. [8, XIII]).

(m) For some of the following statements see e. g. [4] and [17]. Let R be a discrete valuation ring over k with residue field also k, and let R^h be the henselization of R. R^h is again a discrete valuation ring, with residue field k. Denote by K the fraction field of R^h and let $K^{\mathrm{sep}}\supset K$ be a separable algebraic closure of K. Then the integral closure $\overline{R^h}$ of R^h in K^{sep} is still a one-dimensional henselian valuation ring, still has residue field k, and its fraction field is K^{sep}. We denote by the same symbols R, R^h and $\overline{R^h}$ the corresponding schemes (they are not necessarily objects of $\mathbf{Sch}/k$), by η (resp. η^h, $\overline{\eta^h}$) the generic point of R (resp. R^h, $\overline{R^h}$) and by $s=k$ the closed point of all these rings.

If X is any scheme over k we denote its base change along π_R by $X_R=X\times R$, and we abbreviate $(X_R)_\eta$ by X_η when there is no risk of confusion. Similar conventions hold in the case of R^h or $\overline{R^h}$ instead of R, of morphisms and sheaves instead of schemes, and of s, η^h and $\overline{\eta^h}$ instead of η.

(n) Continuing the notation of (m), let $\tilde{X}$ be a scheme over R. Then the *functor of nearby cycles*,

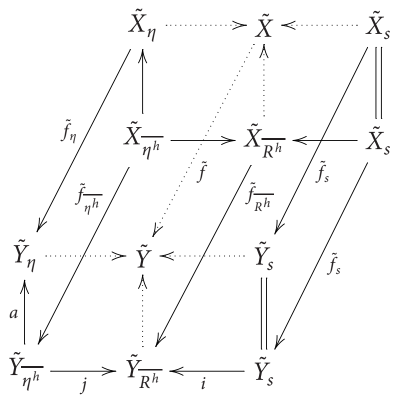
$$\Psi_{\tilde{X}}:\mathfrak{D}^b_{ctf}(\tilde{X}_\eta)\longrightarrow\mathfrak{D}^b_{ctf}(\tilde{X}_s),$$

is explicitly given as follows. Let $\mathscr{F}_\eta\in\mathfrak{D}^b_{ctf}(\tilde{X}_\eta)$.

1. Suppose first that R is henselian. Let $i:\tilde{X}_s\hookrightarrow\tilde{X}_{\overline{R}}$ and $j:\tilde{X}_{\overline{\eta}}\hookrightarrow\tilde{X}_{\overline{R}}$ be the canonical morphisms. Then
$$\Psi_{\tilde{X}}(\mathscr{F}_\eta)=i^*j_*\mathscr{F}_{\overline{\eta}}.$$

2. In the general case, we have
$$\Psi_{\tilde{X}}(\mathscr{F}_\eta)=\Psi_{\tilde{X}_{R^h}}(\mathscr{F}_{\eta^h}),$$
the right hand side being defined as in 1.

(o) Still continuing the above notation, suppose we are given a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ of schemes over R. Then the following diagram is commutative



and we may define the following base change morphisms (the arrows on the back face are denoted by the same symbols as on the front face):

$$\tilde{f}_s^* \Psi_{\tilde{Y}} = \tilde{f}_s^* i^* j_* a^* \xrightarrow{\cong} i^* \tilde{f}_{\overline{R^h}}^* j_* a^* \xrightarrow{(1.12)} i^* j_* \tilde{f}_{\overline{\eta^h}}^* a^* \xrightarrow{\cong} i^* j_* a^* \tilde{f}_\eta^* = \Psi_{\tilde{X}} \tilde{f}_\eta^*, \tag{1.22}$$

$$\tilde{f}_{s!} \Psi_{\tilde{X}} = \tilde{f}_{s!} i^* j_* a^* \xrightarrow[\cong]{(1.13)} i^* \tilde{f}_{\overline{R^h}!} j_* a^* \xrightarrow{(1.16)} i^* j_* \tilde{f}_{\overline{\eta^h}!} a^* \xrightarrow[\cong]{(1.13)} i^* j_* a^* \tilde{f}_{\eta!} = \Psi_{\tilde{Y}} \tilde{f}_{\eta!}, \tag{1.23}$$

$$\Psi_{\tilde{Y}} \tilde{f}_{\eta *} = i^* j_* a^* \tilde{f}_{\eta *} \xrightarrow{(1.12)} i^* j_* \tilde{f}_{\overline{\eta^h} *} a^* \xrightarrow{\cong} i^* \tilde{f}_{\overline{R^h} *} j_* a^* \xrightarrow{(1.12)} \tilde{f}_{s*} i^* j_* a^* = \tilde{f}_{s*} \Psi_{\tilde{X}}, \tag{1.24}$$

$$\Psi_{\tilde{X}} \tilde{f}_\eta^! = i^* j_* a^* \tilde{f}_\eta^! \xrightarrow{(1.15)} i^* j_* \tilde{f}_{\overline{\eta^h}}^! a^* \xrightarrow[\cong]{(1.14)} i^* \tilde{f}_{\overline{R^h}}^! j_* a^* \xrightarrow{(1.15)} \tilde{f}_s^! i^* j_* a^* = \tilde{f}_s^! \Psi_{\tilde{Y}}. \tag{1.25}$$

The latter two may also be described as follows:

$$\Psi_{\tilde{Y}} \tilde{f}_{\eta *} \xrightarrow{\text{adj}} \tilde{f}_{s*} \tilde{f}_s^* \Psi_{\tilde{Y}} \tilde{f}_{\eta *} \xrightarrow{(1.22)} \tilde{f}_{s*} \Psi_{\tilde{X}} \tilde{f}_\eta^* \tilde{f}_{\eta *} \xrightarrow{\text{adj}} \tilde{f}_{s*} \Psi_{\tilde{X}},$$
$$\Psi_{\tilde{X}} \tilde{f}_\eta^! \xrightarrow{\text{adj}} \tilde{f}_s^! \tilde{f}_{s!} \Psi_{\tilde{X}} \tilde{f}_\eta^! \xrightarrow{(1.23)} \tilde{f}_s^! \Psi_{\tilde{Y}} \tilde{f}_{\eta!} \tilde{f}_\eta^! \xrightarrow{\text{adj}} \tilde{f}_s^! \Psi_{\tilde{Y}}.$$

If $\tilde{f}$ is smooth then (1.22) is an isomorphism and if $\tilde{f}$ is étale then (1.25) is its inverse; if $\tilde{f}$ is proper then (1.23) is an isomorphism and (1.24) is its inverse (cf. **(k)**).

It follows from the compatibility with respect to composition of scheme morphisms of the various base change morphisms appearing in the definition above (see **(k)**) that also (1.22) until (1.25) behave well with respect to composition. More precisely, suppose we are given an additional morphism $\tilde{g} : \tilde{Y} \to \tilde{Z}$ of schemes over R. Then the following diagram is commutative:

$$\begin{array}{ccccc} \tilde{f}_s^* \tilde{g}_s^* \Psi_{\tilde{Z}} & \xrightarrow{(1.22)} & \tilde{f}_s^* \Psi_{\tilde{Y}} \tilde{g}_\eta^* & \xrightarrow{(1.22)} & \Psi_{\tilde{X}} \tilde{f}_\eta^* \tilde{g}_\eta^* \\ & \searrow^{\cong} & & \swarrow^{\cong} & \\ & (\tilde{g}\tilde{f})_s^* \Psi_{\tilde{Z}} & \xrightarrow[(1.22)]{} & \Psi_{\tilde{X}} (\tilde{g}\tilde{f})_\eta^* & \end{array}$$

Similar diagrams commute in the other cases.

§5 Filtered derived categories

Our main reference for this paragraph is [15, Chapter V].

(p) Fix an abelian category $\mathbf{A}$. We denote by $\mathfrak{C}(\mathbf{A})$ the category of complexes in $\mathbf{A}$. A (finite) filtration on an object L of $\mathfrak{C}(\mathbf{A})$ is a decreasing sequence of objects in $\mathfrak{C}(\mathbf{A})$,

$$\cdots \supset F^i L \supset F^{i+1} L \supset \cdots, \quad i \in \mathbb{Z},$$

such that $F^i L = 0$ for $i \gg 0$ and $F^i L = L$ for $i \ll 0$. A morphism $f : L \to M$ in $\mathfrak{C}(\mathbf{A})$ is said to preserve the filtrations $(F^i L)$, $(F^i M)$ on L and M, respectively, if $f(F^i L) \subset F^i M$ for all $i \in \mathbb{Z}$; in this case we denote the induced morphism $F^i L \to F^i M$ by $F^i f$. We define $\mathfrak{C}\mathfrak{f}(\mathbf{A})$ to be the category whose objects are objects L of $\mathfrak{C}(\mathbf{A})$ equipped with a (finite) filtration (always denoted $(F^i L)$), and whose morphisms are morphisms in $\mathfrak{C}(\mathbf{A})$ preserving the filtrations. This is an additive category.

Given $-\infty \le a \le b \le \infty$ there are full subcategories $\mathfrak{C}\mathfrak{f}^{[a,b]}(\mathbf{A})$ of $\mathfrak{C}\mathfrak{f}(\mathbf{A})$ consisting of those objects L such that $F^i L = L$ for $i \le a$ and $F^i L = 0$ for $i > b$ (notice the asymmetry). There are truncation functors $\tau^{[a,b]} : \mathfrak{C}\mathfrak{f}(\mathbf{A}) \to \mathfrak{C}\mathfrak{f}(\mathbf{A})$ which are defined on objects by $\tau^{[a,b]}(L) = F^a L / F^{b+1} L$ with the induced filtration $F^i \tau^{[a,b]}(L) = F^i L / F^{b+1} L$ in the interval $[a, b]$ (and in an obvious way on morphisms). We abbreviate $[-\infty, b]$ (resp. $[a, \infty]$) by $\le b$ (resp. $\ge a$).

We denote by ω the forgetful functor $\mathfrak{C}\mathfrak{f}(\mathbf{A}) \to \mathfrak{C}(\mathbf{A})$ and for each $i \in \mathbb{Z}$ by gr^i the composition of functors $\omega\tau^{[i,i]} : \mathfrak{C}\mathfrak{f}(\mathbf{A}) \to \mathfrak{C}(\mathbf{A})$ which maps L to $F^i L / F^{i+1} L$.

(q) We say that a morphism f in $\mathfrak{C}\mathfrak{f}(\mathbf{A})$ is a quasi-isomorphism if $F^i f$ (or, equivalently (cf. [15, V, 1.2]), $\mathrm{gr}^i f$) is a quasi-isomorphism in $\mathfrak{C}(\mathbf{A})$ for all $i \in \mathbb{Z}$. There is a way to define a triangulated homotopy category $\mathfrak{K}\mathfrak{f}(\mathbf{A})$ to which this notion of quasi-isomorphism extends (the details are not important for our purposes; cf. [15, V, p. 271]). We denote by $\mathfrak{D}\mathfrak{f}(\mathbf{A})$ the triangulated category obtained by the usual localization process with respect to these quasi-isomorphisms. Similarly, by starting from $\mathfrak{C}\mathfrak{f}^{[a,b]}(\mathbf{A})$ one obtains categories $\mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{A})$ and these are canonically full subcategories of $\mathfrak{D}\mathfrak{f}(\mathbf{A})$ ([15, V, 1.2.7.1]).

By $\mathfrak{D}^b\mathfrak{f}(\mathbf{A})$ we denote the full subcategory of $\mathfrak{D}\mathfrak{f}(\mathbf{A})$ consisting of those objects L such that $\mathrm{H}^n \mathrm{gr}^i L = 0$ for all but finitely many i and n. If we denote by $\mathfrak{C}^b\mathfrak{f}(\mathbf{A})$ the full subcategory of $\mathfrak{C}\mathfrak{f}(\mathbf{A})$ consisting of those objects L such that $L^n = 0$ for all but finitely many n then, as usual, $\mathfrak{D}^b\mathfrak{f}(\mathbf{A})$ is naturally equivalent to the category arising from $\mathfrak{C}^b\mathfrak{f}(\mathbf{A})$ by localizing with respect to the quasi-isomorphisms. Similar definitions and remarks apply to $\mathfrak{D}^+\mathfrak{f}(\mathbf{A})$ and $\mathfrak{D}^-\mathfrak{f}(\mathbf{A})$.

The functors from (p) induce additive functors $\tau^{[a,b]}$, ω and gr^i between derived categories. Moreover, for every $i \in \mathbb{Z}$, there is a distinguished triangle of functors $\tau^{\ge i} \to \mathbb{1} \to \tau^{\le i-1} \to^+$, i. e. natural transformations of functors which, evaluated at any object of $\mathfrak{D}\mathfrak{f}(\mathbf{A})$, yield a distinguished triangle.

(r) Fix in addition two abelian categories $\mathbf{B}$ and $\mathbf{C}$.

We say that a triangulated functor $\tilde{G} : \mathfrak{D}\mathfrak{f}(\mathbf{A}) \to \mathfrak{D}\mathfrak{f}(\mathbf{B})$ (resp. $\tilde{G} : \mathfrak{D}\mathfrak{f}(\mathbf{A})^\circ \to \mathfrak{D}\mathfrak{f}(\mathbf{B})$) is filtered if

$$\tilde{G}(\mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{A})) \subset \mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{B}) \quad (\text{resp. } \tilde{G}(\mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{A})^\circ) \subset \mathfrak{D}\mathfrak{f}^{[-b,-a]}(\mathbf{B}))$$

for all $a \le b$. A filtered lift of a triangulated functor $G : \mathfrak{D}(\mathbf{A}) \to \mathfrak{D}(\mathbf{B})$ (resp. $G : \mathfrak{D}(\mathbf{A})^\circ \to \mathfrak{D}(\mathbf{B})$) is a pair $(\tilde{G}, \varphi_G)$ where $\tilde{G}$ is a filtered triangulated functor as above and φ_G is an isomorphism of functors $\varphi_G : \omega\tilde{G} \to G\omega$.

Similar definitions can be formulated for bifunctors: We say that a triangulated bifunctor $\tilde{G} : \mathfrak{D}\mathfrak{f}(\mathbf{A}) \times \mathfrak{D}\mathfrak{f}(\mathbf{B}) \to \mathfrak{D}\mathfrak{f}(\mathbf{C})$ (resp. $\tilde{G} : \mathfrak{D}\mathfrak{f}(\mathbf{A})^\circ \times \mathfrak{D}\mathfrak{f}(\mathbf{B}) \to \mathfrak{D}\mathfrak{f}(\mathbf{C})$) is filtered if

$$\tilde{G}\left(\mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{A}) \times \mathfrak{D}\mathfrak{f}^{[c,d]}(\mathbf{B})\right) \subset \mathfrak{D}\mathfrak{f}^{[a+c,b+d]}(\mathbf{C})$$

$$(\text{resp. } \tilde{G}\left(\mathfrak{D}\mathfrak{f}^{[a,b]}(\mathbf{A})^\circ \times \mathfrak{D}\mathfrak{f}^{[c,d]}(\mathbf{B})\right) \subset \mathfrak{D}\mathfrak{f}^{[-b+c,-a+d]}(\mathbf{C})).$$

A filtered lift of a triangulated bifunctor $G : \mathfrak{D}(\mathbf{A}) \times \mathfrak{D}(\mathbf{B}) \to \mathfrak{D}(\mathbf{C})$ (resp. $G : \mathfrak{D}(\mathbf{A})^{\circ} \times \mathfrak{D}(\mathbf{B}) \to \mathfrak{D}(\mathbf{C})$) is a pair $(\tilde{G}, \varphi_G)$ where $\tilde{G}$ is a filtered bifunctor as above and φ_G is an isomorphism of bifunctors $\varphi_G : \omega\tilde{G} \to G(\omega \times \omega)$.

Finally, by a filtered lift of a morphism $H : G_1 \to G_2$ between triangulated functors (resp. bifunctors) we mean a morphism $\tilde{H} : \tilde{G}_1 \to \tilde{G}_2$ between filtered lifts of G_1 and G_2 such that the following diagram commutes:

$$\begin{array}{ccc} \omega\tilde{G}_1 & \xrightarrow{\omega\tilde{H}} & \omega\tilde{G}_2 \\ {\scriptstyle\varphi_{G_1}}\downarrow & & \downarrow{\scriptstyle\varphi_{G_2}} \\ G_1\omega & \xrightarrow[H\omega]{} & G_2\omega \end{array} \qquad \text{respectively} \qquad \begin{array}{ccc} \omega\tilde{G}_1 & \xrightarrow{\omega\tilde{H}} & \omega\tilde{G}_2 \\ {\scriptstyle\varphi_{G_1}}\downarrow & & \downarrow{\scriptstyle\varphi_{G_2}} \\ G_1(\omega \times \omega) & \xrightarrow[H(\omega\times\omega)]{} & G_2(\omega \times \omega) \end{array}$$

(s) Assume that $\mathbf{A}$ has enough injectives. Then, by [15, V, 1.4.5] the bifunctor $\mathbf{R}\mathrm{Hom} : \mathfrak{D}(\mathbf{A})^{\circ} \times \mathfrak{D}^{+}(\mathbf{A}) \to \mathfrak{D}(\mathbf{Ab})$ (where $\mathbf{Ab}$ denotes the category of abelian groups) possesses a filtered lift $\widetilde{\mathbf{R}\mathrm{Hom}} : \mathfrak{D}\mathfrak{f}(\mathbf{A})^{\circ} \times \mathfrak{D}\mathfrak{f}^{+}(\mathbf{A}) \to \mathfrak{D}\mathfrak{f}(\mathbf{Ab})$. Moreover, there is a natural isomorphism of bifunctors

$$\mathrm{Hom} \xrightarrow{\cong} \mathrm{H}^{0}\, \omega\tau^{\geq 0}\widetilde{\mathbf{R}\mathrm{Hom}},$$

where the left hand side denotes the bifunctor of filtered morphisms ([15, V, 1.4.6]).

Specializing to the case of interest, let $f : X \to Y$ be a morphism of schemes and set $\mathbf{A}$ to be the category of sheaves of Λ-modules for the étale topology on X. According to [15, V, 2], the (bi)functors f_*, f^*, $\mathbf{R}\underline{\mathrm{Hom}}$ and $\otimes$ on $\mathfrak{D}^{?}(X, \Lambda)$ etc. possess natural filtered lifts to $\mathfrak{D}^{?}\mathfrak{f}(X) := \mathfrak{D}^{?}\mathfrak{f}(\mathbf{A})$ etc.:

$$\begin{aligned} \tilde{f}_* &: \mathfrak{D}^{+}\mathfrak{f}(X) \to \mathfrak{D}^{+}\mathfrak{f}(Y), \\ \tilde{f}^* &: \mathfrak{D}^{-}\mathfrak{f}(Y) \to \mathfrak{D}^{-}\mathfrak{f}(X), \\ \widetilde{\mathbf{R}\underline{\mathrm{Hom}}} &: \mathfrak{D}\mathfrak{f}(X)^{\circ} \times \mathfrak{D}^{+}\mathfrak{f}(X) \to \mathfrak{D}\mathfrak{f}(X), \\ \tilde{\otimes} &: \mathfrak{D}^{-}\mathfrak{f}(X) \times \mathfrak{D}\mathfrak{f}(X) \to \mathfrak{D}\mathfrak{f}(X). \end{aligned}$$

Moreover, they enjoy the same adjointness relations as the original functors.

The global sections functor $\Gamma(X, -) : \mathfrak{D}^{+}(X) \to \mathfrak{D}^{+}(\mathbf{Ab})$ possesses a filtered lift

$$\tilde{\Gamma}(X, -) : \mathfrak{D}^{+}\mathfrak{f}(X) \longrightarrow \mathfrak{D}^{+}\mathfrak{f}(\mathbf{Ab})$$

according to [15, V, 2.2.7]. There is a canonical isomorphism

$$\tilde{\Gamma}(X, \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}) \xrightarrow{\cong} \widetilde{\mathbf{R}\mathrm{Hom}}$$

of filtered bifunctors $\mathfrak{D}^{-}(X)^{\circ} \times \mathfrak{D}^{+}(X) \to \mathfrak{D}^{+}(\mathbf{Ab})$ ([15, V, 2.2.10]). In particular, there is a canonical isomorphism of bifunctors

$$\mathrm{H}^{0}(X, \omega\tau^{\geq 0}\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}) \cong \mathrm{Hom}\,.$$

2 Cohomological correspondences and operations on them

§1 Correspondences

To define our main object of study, cohomological correspondences, we first need the notion of a correspondence.

Definition 2.1 1. Let X_1, X_2 be two schemes. A *correspondence (from X_1 to X_2)* is a morphism $c : C \to X_1 \times X_2$.[1] It is called a *self-correspondence* if $X_1 = X_2$.

2. Given two correspondences $c = (c_1, c_2) : C \to X_1 \times X_2$, $b = (b_1, b_2) : B \to Y_1 \times Y_2$, a *morphism from c to b* is a triple $[f] = (f_1, f^\natural, f_2)$ of morphisms of schemes making the following diagram commutative:

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f^\natural} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xleftarrow[b_1]{} & B & \xrightarrow[b_2]{} & Y_2 \end{array} \tag{2.1}$$

A *morphism of self-correspondences* in addition satisfies $f_1 = f_2$.

This defines a category in an obvious way, the category $\mathbf{Cor}(k)$ of correspondences over k. It has a terminal object, namely the trivial correspondence $k \to k \times k$, denoted c_k. Given a correspondence $c : C \to X_1 \times X_2$, the structure morphism $c \to c_k$ is denoted $[\pi]_c$. Similarly one defines the category $\mathbf{Cor}(R)$ of correspondences over a discrete valuation ring R.

The subcategories of self-correspondences are denoted $\mathbf{sCor}(k)$, $\mathbf{sCor}(R)$, respectively.

Notation 2.2 Whenever we mention a correspondence $c : C \to X_1 \times X_2$ in the sequel and if not explicitly stated otherwise, we will tacitly assume the following notational convention: $p_i : X_1 \times X_2 \to X_i$ denotes the projection onto the i^{th} factor, $c_i = p_i \circ c$. Similarly, if $[f]$ is a morphism of correspondences we will assume $[f] = (f_1, f^\natural, f_2)$ except if mentioned otherwise.

Since correspondences are built out of schemes and morphisms of schemes we may transfer several notions from scheme theory to the theory of correspondences as follows.

Definition 2.3 Let $\mathscr{P}$ be a property of morphisms of schemes.

1. A morphism $(f_1, f^\natural, f_2)$ between correspondences is said to possess the property $\mathscr{P}$ if each component $f_1, f^\natural, f_2$ does.

2. A correspondence c is said to possess the property $\mathscr{P}$ if the structure morphism $[\pi]_c$ does.

Example 2.4 1. Let $c : C \to X_1 \times X_2$ be a correspondence and $j : W \hookrightarrow C$ an open (resp. closed) subscheme. We may restrict c to W, denoted $c|_W$, and thus obtain an open (resp. closed) immersion $[j_W] = (j, \mathbb{1}_{X_1}, \mathbb{1}_{X_2}) : c|_W \hookrightarrow c$.

2. Similarly, let $c : C \to X_1 \times X_2$ be a correspondence and $j_i : U_i \hookrightarrow X_i$ open subschemes, $i = 1, 2$. Then we may restrict c to

$$c|^{U_1, U_2} : c_1^{-1}(U_1) \cap c_2^{-1}(U_2) \longrightarrow U_1 \times U_2.$$

[1]Recall (1.(a)) that this means that C, X_1, X_2 are separated schemes of finite type over k, $X_1 \times X_2$ is actually $X_1 \times_k X_2$ and c is a morphism over k.

Again, there is an open immersion $[j^{U_1,U_2}] = (i, j_1, j_2) : c|^{U_1,U_2} \hookrightarrow c$, where $i : c_1^{-1}(U_1) \cap c_2^{-1}(U_2) \hookrightarrow C$.

If $X_1 = X_2 = X$ and $U_1 = U_2 = U$ we will simply write $c|^U$ and $[j^U]$ instead of $c|^{U,U}$ and $[j^{U,U}]$, respectively.

3. Finally, let $c : C \to X_1 \times X_2$ be a correspondence, let $Z_i \subset X_i$ be closed subschemes, and assume that c restricts to a correspondence

$$c|^{Z_1,Z_2} : c_2^{-1}(Z_2)_{\mathrm{red}} \longrightarrow Z_1 \times Z_2.$$

There is an obvious closed immersion $c|^{Z_1,Z_2} \hookrightarrow c$, denoted $[i^{Z_1,Z_2}]$. Again, if $X_1 = X_2$ and $Z_1 = Z_2 = Z$ we will simply write $c|^Z$ and $[i^Z]$.

Note the ambiguity if $Z_i = U_i \subset X_i$ are open as well as closed subschemes. Then, in general, $c|^{Z_1,Z_2}$ as defined now is not equal to $c|^{U_1,U_2}$ as defined in 2. Context will always make clear which one is meant.

Note also that the condition for c restricting to $c|^Z$ is equivalent to $c_1(c_2^{-1}(Z)) \subset Z$ (set-theoretically). In a later section we will say that Z is "c-invariant" if this condition is satisfied and it will play an important role in the course of the argument (cf. 4.§1).

We will be interested in the second part of the above definition only when $\mathscr{P}$ is the property of being proper. In fact, this case will be quite important in the sequel so we end this paragraph with a simple but useful observation regarding it. Notice that Definition 2.3 tells us in particular what a compactification of a correspondence is and what it means for a morphism of correspondences to extend another one.

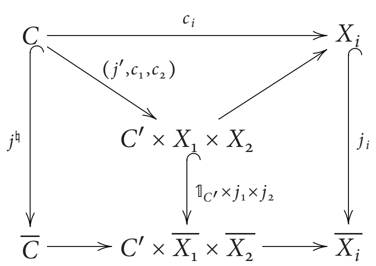
Lemma 2.5 *1. Every correspondence admits a compactification. In fact, more is true: Let $c = (c_1, c_2) : C \to X_1 \times X_2$ be a correspondence and $j_1 : X_1 \hookrightarrow \overline{X_1}$ and $j_2 : X_2 \hookrightarrow \overline{X_2}$ any two given compactifications. Then there exists a compactification $j^\natural : C \hookrightarrow \overline{C}$ and a correspondence $\overline{c} : \overline{C} \to \overline{X_1} \times \overline{X_2}$ such that the pair $(\overline{c}, (j_1, j^\natural, j_2))$ is a compactification of c.*

2. Given a morphism of correspondences $[f] : c \to b$ there exist compactifications $\overline{c}$ of c and $\overline{b}$ of b and a morphism of correspondences $[\overline{f}] : \overline{c} \to \overline{b}$ extending $[f]$.

Proof 1. Choose any compactification $j' : C \hookrightarrow C'$ and let $\overline{C}$ be the closure of the image of the embedding

$$j^\natural = (j', j_1 \circ c_1, j_2 \circ c_2) : C \longrightarrow C' \times \overline{X_1} \times \overline{X_2}.$$

The composition $\overline{C} \to C' \times \overline{X_1} \times \overline{X_2} \to \overline{X_1} \times \overline{X_2}$ with the projection defines $\overline{c}$ and $[j] = (j_1, j^\natural, j_2)$ is a morphism since the following diagram clearly commutes for $i = 1, 2$:



Finally, $j^\natural$ and hence $[j]$ are compactifications.

2. Assume that $c : C \to X_1 \times X_2$ and $b : D \to Y_1 \times Y_2$ are the two correspondences. First choose a compactification $[j] : b \hookrightarrow \overline{b}$ as in part 1, where $\overline{b} : \overline{D} \to \overline{Y}_1 \times \overline{Y}_2$ is a proper correspondence. Next, for $i = 1, 2$, let $k_i : X_i \hookrightarrow \overline{X}_i$ be a compactification such that f_i extends to a morphism $\overline{f}_i : \overline{X}_i \to \overline{Y}_i$. Finally, choose a compactification $k^\natural : C \hookrightarrow \overline{C}$ such that the morphism $(f^\natural, c) : C \to D \times_{Y_1 \times Y_2} (X_1 \times X_2)$ extends to a morphism $(\overline{f}^\natural, \overline{c}) : \overline{C} \to \overline{D} \times_{\overline{Y}_1 \times \overline{Y}_2} (\overline{X}_1 \times \overline{X}_2)$. With similar diagrams as in part 1, one checks easily that $(k_1, k^\natural, k_2) : c \to \overline{c}$ is a morphism of correspondences (hence clearly a compactification), that $[\overline{f}] = (\overline{f}_1, \overline{f}^\natural, \overline{f}_2)$ is a morphism of correspondences, and that it extends f. □

§2 Cohomological correspondences

Now, we may introduce our main object of study.

Definition 2.6 Let $c = (c_1, c_2) : C \to X_1 \times X_2$ be a correspondence and let $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$, $i = 1, 2$ (see 1.(d)). A *cohomological correspondence (from $\mathscr{F}_1$ to $\mathscr{F}_2$ lifting c)* is a morphism

$$u : c_{2!} c_1^* \mathscr{F}_1 \to \mathscr{F}_2.$$ [2]

The set of cohomological correspondences from $\mathscr{F}_1$ to $\mathscr{F}_2$ lifting c will be denoted $\mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$.

§3 Pushforward

In the notation of 2.1, assume one of the following conditions is satisfied:

(F1) The left square of (2.1) is cartesian.

(F2) The morphisms f_1 and $f^\natural$ are both proper.

(F3) The morphisms c_1 and b_1 are both proper.

In each of these cases we will define a base change morphism of functors

$$b_1^* f_{1!} \longrightarrow f^\natural_! c_1^*. \tag{2.2}$$

In the cases (F1) and (F2) this is simply (1.13) and (1.12), respectively. In the case (F3) it is defined as the composition

$$b_1^* f_{1!} \xrightarrow{\text{adj}} b_1^* f_{1!} c_{1*} c_1^* = b_1^* f_{1!} c_{1!} c_1^* \xrightarrow{\cong} b_1^* b_{1!} f^\natural_! c_1^* = b_1^* b_{1*} f^\natural_! c_1^* \xrightarrow{\text{adj}} f^\natural_! c_1^*. \tag{2.3}$$

Or, alternatively, it arises by adjointness from the base change morphism

$$c_{1!} f^{\natural !} \xrightarrow{\text{bc (1.11)}} f_1^! b_{1!}$$

via the identifications $c_{1*} = c_{1!}$ and $b_{1*} = b_{1!}$.

Definition 2.7 In the notation of 2.1, suppose (F1) (resp. (F2), (F3)) is satisfied. We define the proper pushforward map (of type (F1) (resp. (F2), (F3))) associated to $[f]$,

$$[f]_! : \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_b(f_{1!}\mathscr{F}_1, f_{2!}\mathscr{F}_2),$$

[2] This definition is slightly different from the one in [14, III, 3.2]; they coincide if c is proper.

as follows. Given $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, let $[f]_!(u)$ be the composition

$$[f]_!(u) : b_{2!}b_1^*(f_{1!}\mathscr{F}_1) \xrightarrow{(2.2)} b_{2!}f_!^{\natural}c_1^*\mathscr{F}_1 \xrightarrow{\cong} f_{2!}c_{2!}c_1^*\mathscr{F}_1 \xrightarrow{f_{2!}u} f_{2!}\mathscr{F}_2.$$

$[f]_!(u)$ is called the *proper pushforward of u along* $[f]$ *(of type ...)*.

Lemma 2.8 *Proper pushforwards are compatible with composition in the following sense. Let* $[f] : a \to b$ *and* $[g] : b \to c$ *be two morphisms of correspondences such that in both cases (Fj) is satisfied. Then (Fj) is satisfied in the case* $[g][f] : a \to c$ *also and the following diagram commutes for any* $\mathscr{F}_i$*:*

$$\begin{array}{ccc}
\mathrm{Hom}_a(\mathscr{F}_1, \mathscr{F}_2) & \xrightarrow{([g][f])_!} & \mathrm{Hom}_c((g_1f_1)_!\mathscr{F}_1, (g_2f_2)_!\mathscr{F}_2) \\
{\scriptstyle [f]_!}\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathrm{Hom}_b(f_{1!}\mathscr{F}_1, f_{2!}\mathscr{F}_2) & \xrightarrow[{[g]_!}]{} & \mathrm{Hom}_c(g_{1!}f_{1!}\mathscr{F}_1, g_{2!}f_{2!}\mathscr{F}_2)
\end{array}$$

Proof It is clear that (Fj) is satisfied in the case $[g][f]$. Now, fix $u \in \mathrm{Hom}_a(\mathscr{F}_1, \mathscr{F}_2)$ and consider the following diagram (where all unlabeled isomorphisms are the obvious ones):

$$\begin{array}{ccccccc}
c_{2!}c_1^*(g_1f_1)_!\mathscr{F}_1 & \xrightarrow{(2.2)} & c_{2!}(g^{\natural}f^{\natural})_!a_1^*\mathscr{F}_1 & \xrightarrow{\cong} & (g_2f_2)_!a_{2!}a_1^*\mathscr{F}_1 & \xrightarrow{(g_2f_2)_!u} & (g_2f_2)_!\mathscr{F}_2 \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
c_{2!}c_1^*g_{1!}f_{1!}\mathscr{F}_1 & (1) & c_{2!}g_!^{\natural}f_!^{\natural}a_1^*\mathscr{F}_1 & \xrightarrow{\cong} & g_{2!}f_{2!}a_{2!}a_1^*\mathscr{F}_1 & \xrightarrow{g_{2!}f_{2!}u} & g_{2!}f_{2!}\mathscr{F}_2 \\
\downarrow{\scriptstyle(2.2)} & & \| & & \| & & \| \\
c_{2!}g_!^{\natural}b_1^*f_{1!}\mathscr{F}_1 & \xrightarrow[(2.2)]{} & c_{2!}g_!^{\natural}f_!^{\natural}a_1^*\mathscr{F}_1 & & & & \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \\
g_{2!}b_{2!}b_1^*f_{1!}\mathscr{F}_1 & \xrightarrow[(2.2)]{} & g_{2!}b_{2!}f_!^{\natural}a_1^*\mathscr{F}_1 & \xrightarrow{\cong} & g_{2!}f_{2!}a_{2!}a_1^*\mathscr{F}_1 & \xrightarrow{g_{2!}f_{2!}u} & g_{2!}f_{2!}\mathscr{F}_2
\end{array}$$

The top row is $([g][f])_!u$, the bottom row $g_{2!}([f]_!u)$, hence commutativity of the outer rectangle would imply the lemma.

All inner squares except (1) are obviously commutative. To prove the lemma, we thus have to show the commutativity of (1), i. e. of the following diagram:

$$\begin{array}{ccccc}
 & c_1^*(g_1f_1)_! & \xrightarrow{(2.2)} & (g^{\natural}f^{\natural})_!a_1^* & \\
 \swarrow{\scriptstyle\cong} & & & & \searrow{\scriptstyle\cong} \\
c_1^*g_{1!}f_{1!} & \xrightarrow[(2.2)]{} & g_!^{\natural}b_1^*f_{1!} & \xrightarrow[(2.2)]{} & g_!^{\natural}f_!^{\natural}a_1^*
\end{array} \tag{2.4}$$

But this follows from the fact that the base change morphism (2.2) is compatible with composition (1.(k)). □

We will now apply the pushforward construction in the case of the structure morphism. Recall that $\mathbf{R}\Gamma_c(X, -)$ denotes the functor $\pi_{X!}$ for any scheme X (see 1.(e)).

Definition 2.9 Let $c : C \to X_1 \times X_2$ be a correspondence and assume that c_1 is proper. Then (F3) is satisfied for the structure morphism $[\pi]_c$ and we may define, for any $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, the proper pushforward of u along $[\pi]_c$:

$$\mathbf{R}\Gamma_c(u) := [\pi]_{c!}(u) : \mathbf{R}\Gamma_c(X_1, \mathscr{F}_1) \longrightarrow \mathbf{R}\Gamma_c(X_2, \mathscr{F}_2).$$

Remark 2.10 In the notation of 2.9, let $(\overline{c} : \overline{C} \to \overline{X}_1 \times \overline{X}_2, [j])$ be a compactification of c. Then $\pi_{\overline{X}_1} \overline{c}_1 = \pi_{\overline{C}}$ is proper hence so is $\overline{c}_1$. This implies that $[j]$ satisfies (F3) and $\overline{u} := [j]_!(u) \in \mathrm{Hom}_{\overline{c}}(j_{1!}\mathscr{F}_1, j_{2!}\mathscr{F}_2)$ is defined. We may then ask what the relationship is between $\mathbf{R}\Gamma_c(u)$ and $\mathbf{R}\Gamma_c(\overline{u})$. Of course, it is the nicest possible, namely the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{R}\Gamma_c(X_1, \mathscr{F}_1) & \xrightarrow{\mathbf{R}\Gamma_c(u)} & \mathbf{R}\Gamma_c(X_2, \mathscr{F}_2) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathbf{R}\Gamma_c(\overline{X}_1, j_{1!}\mathscr{F}_1) & \xrightarrow[\mathbf{R}\Gamma_c(\overline{u})]{} & \mathbf{R}\Gamma_c(\overline{X}_2, j_{2!}\mathscr{F}_2)
\end{array}$$

(Here the vertical arrows are induced by the isomorphisms $(\pi_{\overline{X}_i} j_i)_! \xrightarrow{\cong} \pi_{\overline{X}_i!} j_{i!}$.) Indeed, this is a special case of 2.8.

Notice also that for the commutativity of the diagram above we did not use the fact that $[j]$ is a compactification. It would have sufficed to assume that $[j]$ is an open immersion into a correspondence $\overline{c}$ with $\overline{c}_1$ proper.

§4 Pullback

In the notation of 2.1 assume one of the following conditions is satisfied:

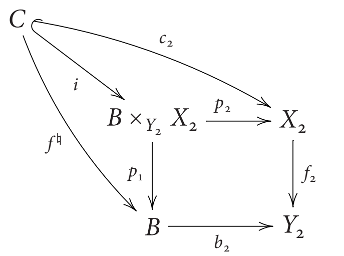
(B1) The canonical map $C \to B \times_{Y_2} X_2$ induces an isomorphism on the reduced subschemes.

(B2) $f^\natural$ and f_2 are both étale.

In both cases there is a base change morphism

$$c_{2!} f^{\natural *} \longrightarrow f_2^* b_{2!}. \tag{2.5}$$

Namely, in the case (B2) this is simply (1.11), while in the case (B1) it is defined as follows. In the commutative diagram



i is a closed immersion onto by assumption. This implies that the adjunction map $\mathrm{adj} : \mathbb{1} \to i_* i^*$ is an isomorphism hence we may define (2.5) by

$$c_{2!} f^{\natural *} \xrightarrow{\cong} p_{2!} i_* i^* p_1^* \xrightarrow[\cong]{\mathrm{adj}^{-1}} p_{2!} p_1^* \xrightarrow[\cong]{(1.13)} f_2^* b_{2!}.$$

Definition 2.11 In the notation of 2.1 assume (B1) (resp. (B2)) is satisfied. We define the pullback map associated to $[f]$ (of type (B1), (B2) resp.),

$$[f]^* : \mathrm{Hom}_b(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_c(f_1^* \mathscr{F}_1, f_2^* \mathscr{F}_2),$$

as follows. Given $u \in \mathrm{Hom}_b(\mathscr{F}_1, \mathscr{F}_2)$, let $[f]^*(u)$ be the composition

$$[f]^*(u) : c_{2!} c_1^* (f_1^* \mathscr{F}_1) \xrightarrow{\cong} c_{2!} f^{\natural *} b_1^* \mathscr{F}_1 \xrightarrow{(2.5)} f_2^* b_{2!} b_1^* \mathscr{F}_1 \xrightarrow{f_2^* u} f_2^* \mathscr{F}_2.$$

$[f]^*(u)$ is called the *pullback of u along $[f]$ (of type ...)*.

Remark 2.12 As in the case of the pushforward construction, the pullback of cohomological correspondences is compatible with composition in the following sense. Let $[f] : a \to b$ and $[g] : b \to c$ be two morphisms of correspondences such that in both cases (Bj) is satisfied. Then (Bj) is satisfied in the case $[g][f] : a \to c$ also and the following diagram commutes for any $\mathscr{F}_i$:

$$\begin{array}{ccc}
\mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) & \xrightarrow{([g][f])^*} & \mathrm{Hom}_a((g_1 f_1)^* \mathscr{F}_1, (g_2 f_2)^* \mathscr{F}_2) \\
{\scriptstyle [g]^*}\downarrow & & \downarrow{\scriptstyle \cong} \\
\mathrm{Hom}_b(g_1^* \mathscr{F}_1, g_2^* \mathscr{F}_2) & \xrightarrow[{[f]^*}]{} & \mathrm{Hom}_a(f_1^* g_1^* \mathscr{F}_1, f_2^* g_2^* \mathscr{F}_2)
\end{array}$$

The proof of this statement is similar to the one of 2.8.

We will now apply this construction to some particular situations described in 2.4.

Example 2.13

1. As in 2.4, let $c : C \to X_1 \times X_2$ be a correspondence and $W \subset C$ an open subset. Then the pullback map (of type (B2))

$$[j_W]^* : \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_{c|_W}(\mathscr{F}_1, \mathscr{F}_2)$$

defines, for any $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, $[j_W]^*(u)$, the *restriction of u to W*. We also denote it by $u|_W$.

2. Similarly, if $c : C \to X_1 \times X_2$ is a correspondence and $j_i : U_i \subset X_i$ are open subschemes, then the pullback map (of type (B2))

$$[j^{U_1, U_2}]^* : \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_{c|^{U_1, U_2}}(\mathscr{F}_1|_{U_1}, \mathscr{F}_2|_{U_2})$$

defines, for any $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, $[j^{U_1, U_2}]^*(u)$, the *restriction of u to U_1, U_2*. We also denote it by $u|^{U_1, U_2}$. If $X_1 = X_2 = X$ and $U_1 = U_2 = U$ we will simply write $u|^U$.

3. Finally, let $c : C \to X_1 \times X_2$ be a correspondence, let $Z_i \subset X_i$ be closed subschemes and assume that $c|^{Z_1, Z_2}$ exists. Then the closed immersion $[i^{Z_1, Z_2}]$ clearly satisfies condition (B1) and we may define the pullback map

$$[i^{Z_1, Z_2}]^* : \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_{c|^{Z_1, Z_2}}(\mathscr{F}_1|_{Z_1}, \mathscr{F}_2|_{Z_2}).$$

If $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, $[i^{Z_1, Z_2}]^*(u)$ is called *the restriction of u to Z_1, Z_2* and it is also denoted $u|^{Z_1, Z_2}$. If $X_1 = X_2$ and $Z_1 = Z_2 = Z$ we will simply write $u|^Z$.

Again, there is an ambiguity if $U_i = Z_i \subset X_i$ are open as well closed subschemes but we will always make clear which one is meant.

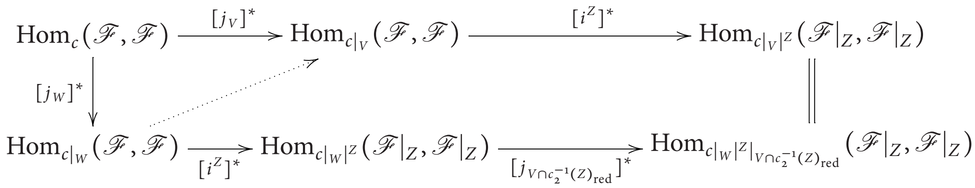
We end this paragraph with a few simple results on composing different types of restriction and pushforward maps.

Lemma 2.14 *Let $c : C \to X \times X$ be a correspondence, let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$, and let $V \subset W \subset C$ be open subsets. Also, let $Z \subset X$ be a closed subscheme and assume that $c|_W|^Z$ exists. Then $c|_V|^Z$ also exists and for every $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$, the following equality holds:*

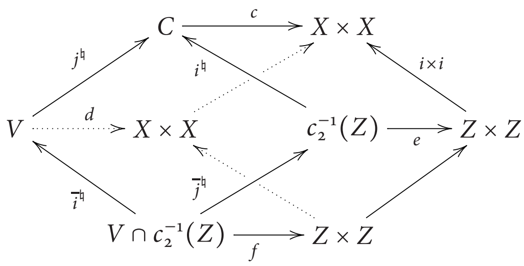
$$u|_V|^Z = u|_W|^Z|_{V \cap c_2^{-1}(Z)_{red}}. \tag{2.6}$$

Notice that the statement is meaningful because the left hand side of (2.6) is a cohomological correspondence from $\mathscr{F}|_Z$ to itself lifting $c|_V|^Z$ while the right hand side is one from $\mathscr{F}|_Z$ to itself lifting $c|_W|^Z|_{V\cap c_2^{-1}(Z)_{\mathrm{red}}}$, and these two correspondences are the same (both are c restricted to $V \cap c_2^{-1}(Z)_{\mathrm{red}} \to Z \times Z$).

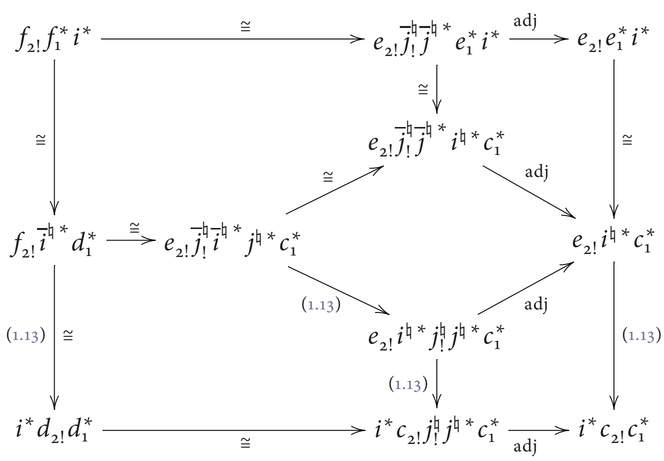
Proof We have to prove that the following rectangle commutes:



By 2.12, $[j_V]^*$ factors through $[j_W]^*$ as indicated by the dotted arrow. Hence we may assume $W = C$ and that $c|^Z$ exists. For notational convenience only we will also assume that $c_2^{-1}(Z) \subset C$ is already reduced. We fix our notation as in the following "cartesian square of correspondences":



We deduce the following diagram:



The parallelogram in the middle commutes by the definition of (1.13), the lower pentagon on the left commutes by the compatibility of (1.13) with respect to composition, the commutativity of the upper pentagon on the left follows immediately from the "cocycle condition" (1.**(d)**), while the two trapezoids on the right clearly commute. Hence the whole diagram is commutative. But applying this diagram to $\mathscr{F}$ and following the two outer paths followed by i^*u yields exactly the left (resp. right) hand side of (2.6). □

Lemma 2.15 *Let $c : C \to X_1 \times X_2$ be a correspondence, let $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$, and let $Z_i \subset X_i$ be closed subschemes contained in open subsets $U_i \subset X_i$. Assume that $c|^{Z_1,Z_2}$ exists. Then $c|^{U_1,U_2}|^{Z_1,Z_2}$ exists as well and for every $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$, the following equality holds:*

$$u|^{Z_1,Z_2}|_{c_1^{-1}(U_1)\cap c_2^{-1}(Z_2)_{red}} = u|^{U_1,U_2}|^{Z_1,Z_2}, \tag{2.7}$$

both sides considered as elements of $\mathrm{Hom}_{c|^{Z_1,Z_2}|_{c_1^{-1}(U_1)\cap c_2^{-1}(Z_2)_{red}}}(\mathscr{F}_1|_{Z_1}, \mathscr{F}_2|_{Z_2})$.

Proof $c|^{Z_1,Z_2}$ exists iff $c_1(c_2^{-1}(Z_2))$ is set-theoretically contained in Z_1 while $c|^{U_1,U_2}|^{Z_1,Z_2}$ exists iff $c_1(c_2^{-1}(Z_2) \cap c_1^{-1}(U_1))$ is set-theoretically contained in Z_1. Hence the first claim is clear.

Essentially the same proof as in the previous lemma shows that the left hand side of (2.7) is equal to $u|_{c_1^{-1}(U_1)}|^{Z_1,Z_2}$, while the right hand side is equal to $u|_{c_1^{-1}(U_1)}|^{U_1,U_2}|^{Z_1,Z_2}$ by 2.12. Setting $u' = u|_{c_1^{-1}(U_1)}$ we thus have to prove

$$u'|^{Z_1,Z_2} = u'|^{U_1,U_2}|^{Z_1,Z_2}. \tag{2.8}$$

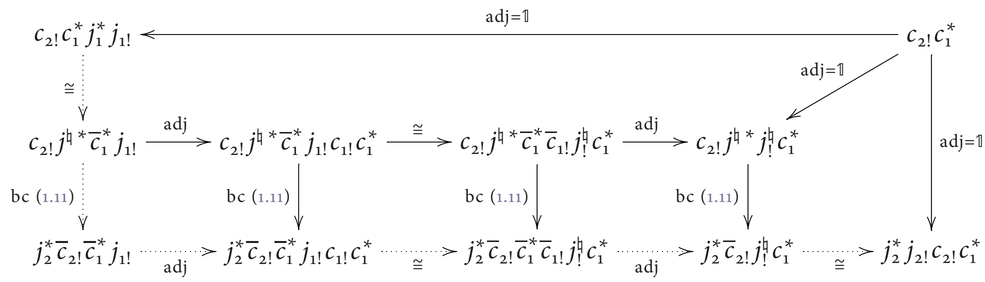
Set $C' = c_1^{-1}(U_1)$, $c' = c|_{C'}$, $d = c'|^{U_1,U_2}$, $e = d|^{Z_1,Z_2}$ and assume for notational convenience again that $c_2'^{-1}(Z_2)$ is reduced. Finally, denote the canonical inclusions by $[j] : d \hookrightarrow c'$, $[i] : e \hookrightarrow d$. Then the claim is equivalent to the commutativity of the following diagram (where we abstain from writing $\mathscr{F}_1$):

$$\begin{array}{ccccccc}
e_{2!}e_1^*i_1^*j_1^* & \overset{\cong}{\dashrightarrow} & e_{2!}i^{\natural *}d_1^*j_1^* & \overset{\mathrm{bc}\ (2.5)}{\dashrightarrow} & i_2^*d_{2!}d_1^*j_1^* & & \\
\Big\downarrow \cong & & \cong\Big\downarrow & & \vdots\ \cong & & \\
 & & e_{2!}i^{\natural *}j^{\natural *}c_1'^* & \overset{\mathrm{bc}\ (2.5)}{\longrightarrow} & i_2^*d_{2!}j^*c_1'^* & \overset{\mathrm{bc}\ (2.5)}{\dashrightarrow} & i_2^*j_2^*c_{2!}'c_1'^* \\
\Big\downarrow & & \cong\Big\downarrow & & & & \Big\downarrow\cong \\
e_{2!}e_1^*(j_1i_1)^* & \underset{\cong}{\longrightarrow} & e_{2!}(j^{\natural}i^{\natural})^*c_1'^* & & \underset{\mathrm{bc}\ (2.5)}{\longrightarrow} & & (j_2i_2)^*c_{2!}'c_1'^*
\end{array}$$

Indeed, the dotted path followed by $i_2^*j_2^*u'$ is equal to the right hand side of (2.8) while the bottom row followed by $(j_2i_2)^*u'$ is equal to the left hand side. Now, the left square of the diagram is commutative by the "cocycle condition", the upper right square is commutative by the naturality of (2.5), and the lower right square commutes because all base change morphisms involved are instances of (1.13) (see 1.**(k)**). □

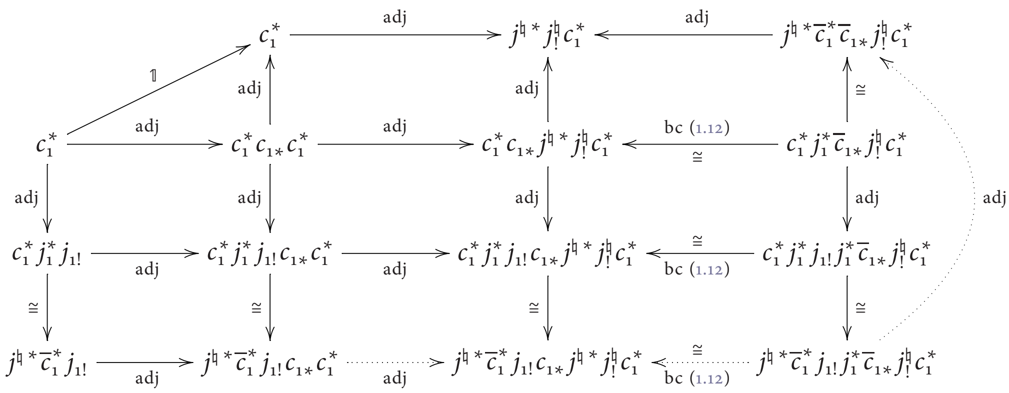
Lemma 2.16 *Let $[j] : c \to \overline{c}$ be an open immersion and assume that c_1 and $\overline{c}_1$ are both proper, and $j^{\natural}$ dominant. Then $[j]^*[j]_!$ is the identity map.*

Proof Consider the following diagram:



Applying the whole diagram to $\mathscr{F}_1$ and following the dotted path by $j_2^*j_{2!}u$ yields $[j]^*[j]_!u$ (any $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$). Hence it suffices to prove the commutativity of the diagram.

The three squares in the lower half commute by the naturality of (1.11), and the commutativity of the trapezoid on the right is easily deduced from the definition of (1.11). For the upper trapezoid we use the description of the transitivity isomorphism $j_{1!}c_{1!} \cong \overline{c}_{1!}j_!^\natural$ given in [2, XVII, 5.1.5]. Since our hypotheses imply that the square $j_1c_1 = \overline{c}_1j^\natural$ is cartesian, it reduces to the dotted path in the following diagram:



This diagram clearly decomposes the upper trapezoid in the previous diagram, and its commutativity follows from the naturality of the morphisms involved. □

§5 Specialization

Let R be a discrete valuation ring as in 1.(m). Recall that we constructed in 1.(n), for each scheme $\tilde{X}$ over R, the nearby cycle functor $\Psi_{\tilde{X}} : \mathfrak{D}^b_{ctf}(\tilde{X}_\eta) \longrightarrow \mathfrak{D}^b_{ctf}(\tilde{X}_s)$. We will use this functor to construct another functor called "specialization functor", similar to the one introduced in [20]. We continue the notation of 1.§4.

Definition 2.17

1. Let X be a scheme over k. A pair $(\tilde{X}, \varphi_X)$ is a *lift of* X if $\tilde{X}$ is an object of $\mathbf{Sch}/R$ and $\varphi = \varphi_X : \tilde{X} \to X_R$ is a morphism in $\mathbf{Sch}/R$ inducing an isomorphism $\varphi_\eta : \tilde{X}_\eta \to X_\eta$. We will sometimes say that $\tilde{X}$ lifts X and let φ_X remain implicit.

2. Let $f : X \to Y$ be a morphism of schemes over k. A triple $(\tilde{f}, \tilde{X}, \tilde{Y})$ is a *lift of* f if $\tilde{X}$ is a lift of X, $\tilde{Y}$ is a lift of Y and $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism in $\mathbf{Sch}/R$ such that $\varphi_Y\tilde{f} = f_R\varphi_X$, i. e. the

following diagram commutes:

$$\begin{array}{ccc} X_R & \xleftarrow{\varphi_X} & \tilde{X} \\ {\scriptstyle f_R}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ Y_R & \xleftarrow{\varphi_Y} & \tilde{Y} \end{array} \tag{2.9}$$

We will also simply say that $\tilde{f} : \tilde{X} \to \tilde{Y}$ lifts f.

3. Given a lift $(\tilde{X}, \varphi)$ of X, we define the *specialization functor (with respect to X, $\tilde{X}$ and φ)*,

$$\mathrm{sp}_{X,\tilde{X},\varphi} : \mathfrak{D}^b_{ctf}(X) \longrightarrow \mathfrak{D}^b_{ctf}(\tilde{X}_s),$$

by $\mathrm{sp}_{X,\tilde{X},\varphi}(\mathscr{F}) = \Psi_{\tilde{X}}(\varphi_\eta^* \mathscr{F}_\eta)$. We will usually write $\mathrm{sp}_{\tilde{X}}$ instead of $\mathrm{sp}_{X,\tilde{X},\varphi}$.

Example 2.18 Let X be a scheme over k. Then clearly $(X_R, \mathbb{1}_{X_R})$ is a lift of X in the above sense. We call it the *trivial lift* of X. Notice that if $f : X \to Y$ is a morphism of schemes then there is a unique lift of f to the trivial lifts X_R and Y_R, namely $f_R : X_R \to Y_R$. f_R is also called the *trivial lift*.

Suppose we are given a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of a morphism $f : X \to Y$ of schemes over k. From the commutativity of (2.9) we deduce the commutativity of

$$\begin{array}{ccccc} X & \xleftarrow{\quad k_X \quad} & X_\eta & \xleftarrow{\quad (\varphi_X)_\eta \quad} & \tilde{X}_\eta \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_\eta}\downarrow & & \downarrow{\scriptstyle \tilde{f}_\eta} \\ Y & \xleftarrow[\quad k_Y \quad]{} & Y_\eta & \xleftarrow[\quad (\varphi_Y)_\eta \quad]{} & \tilde{Y}_\eta \end{array}$$

and we may thus define several base change morphisms (setting $c_X = (\varphi_X)_\eta \circ k_X$ and $c_Y = (\varphi_Y)_\eta \circ k_Y$):

$$\begin{aligned} \mathrm{bc}^* &: \tilde{f}_s^* \mathrm{sp}_{\tilde{Y}} = \tilde{f}_s^* \Psi_{\tilde{Y}} c_Y^* \xrightarrow{(1.22)} \Psi_{\tilde{X}} \tilde{f}_\eta^* c_Y^* \xrightarrow{\cong} \Psi_{\tilde{X}} c_X^* f^* = \mathrm{sp}_{\tilde{X}} f^*, \\ \mathrm{bc}_! &: \tilde{f}_{s!} \mathrm{sp}_{\tilde{X}} = \tilde{f}_{s!} \Psi_{\tilde{X}} c_X^* \xrightarrow{(1.23)} \Psi_{\tilde{Y}} \tilde{f}_{\eta!} c_X^* \xrightarrow[\cong]{(1.13)} \Psi_{\tilde{Y}} c_Y^* f_! = \mathrm{sp}_{\tilde{Y}} f_!, \\ \mathrm{bc}_* &: \mathrm{sp}_{\tilde{Y}} f_* = \Psi_{\tilde{Y}} c_Y^* f_* \xrightarrow{(1.12)} \Psi_{\tilde{Y}} \tilde{f}_{\eta*} c_X^* \xrightarrow{(1.24)} \tilde{f}_{s*} \Psi_{\tilde{X}} c_X^* = \tilde{f}_{s*} \mathrm{sp}_{\tilde{X}}, \\ \mathrm{bc}^! &: \mathrm{sp}_{\tilde{X}} f^! = \Psi_{\tilde{X}} c_X^* f^! \xrightarrow{(1.15)} \Psi_{\tilde{X}} \tilde{f}_\eta^! c_Y^* \xrightarrow{(1.25)} \tilde{f}_s^! \Psi_{\tilde{Y}} c_Y^* = \tilde{f}_s^! \mathrm{sp}_{\tilde{Y}}, \end{aligned}$$

and, as in 1.(0), the latter two can also be described as follows:

$$\begin{aligned} \mathrm{bc}_* &: \mathrm{sp}_{\tilde{Y}} f_* \xrightarrow{\mathrm{adj}} \tilde{f}_{s*} \tilde{f}_s^* \mathrm{sp}_{\tilde{Y}} f_* \xrightarrow{\mathrm{bc}^*} \tilde{f}_{s*} \mathrm{sp}_{\tilde{X}} f^* f_* \xrightarrow{\mathrm{adj}} \tilde{f}_{s*} \mathrm{sp}_{\tilde{X}}, \\ \mathrm{bc}^! &: \mathrm{sp}_{\tilde{X}} f^! \xrightarrow{\mathrm{adj}} \tilde{f}_s^! \tilde{f}_{s!} \mathrm{sp}_{\tilde{X}} f^! \xrightarrow{\mathrm{bc}_!} \tilde{f}_s^! \mathrm{sp}_{\tilde{Y}} f_! f^! \xrightarrow{\mathrm{adj}} \tilde{f}_s^! \mathrm{sp}_{\tilde{Y}}. \end{aligned}$$

Remark 2.19 1. It follows from what was said in 1.(0) that if $\tilde{f}$ is smooth then bc^* is an isomorphism and if $\tilde{f}$ is étale then $\mathrm{bc}^!$ is its inverse. Similarly, if $\tilde{f}$ is proper then bc_* is an isomorphism and $\mathrm{bc}_!$ is its inverse.

2. It also follows from 1.(o) that the base change morphisms just defined are compatible with composition in the following sense. Suppose we are given an additional lift $\tilde{g} : \tilde{Y} \to \tilde{Z}$ of a morphism $g : Y \to Z$ of schemes over k. Then the following diagram commutes:

$$\begin{array}{ccccc} \tilde{f}_s^* \tilde{g}_s^* \mathrm{sp}_{\tilde{Z}} & \xrightarrow{\mathrm{bc}^*} & \tilde{f}_s^* \mathrm{sp}_{\tilde{Y}} g^* & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{\tilde{X}} f^* g^* \\ & \cong \searrow & & \swarrow \cong & \\ & (\tilde{g}\tilde{f})_s^* \mathrm{sp}_{\tilde{Z}} & \xrightarrow[\mathrm{bc}^*]{} & \mathrm{sp}_{\tilde{X}} (gf)^* & \end{array}$$

Similar diagrams commute in the case of the other base change morphisms.

3. There is also a noteworthy compatibility between the different base change morphisms introduced above. Suppose we are given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ X & \xrightarrow[g']{} & W \end{array} \qquad \text{and lifts} \qquad \begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{Z} \\ {\scriptstyle \tilde{f}}\uparrow & & \uparrow{\scriptstyle \tilde{f}'} \\ \tilde{X} & \xrightarrow[\tilde{g}']{} & \tilde{W} \end{array}$$

and assume that both $\tilde{g}$ and $\tilde{g}'$ are proper. Then the following diagram commutes:

$$\begin{array}{ccc} \tilde{f}_s'^* \mathrm{sp}_{\tilde{Z}} g_! & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{\tilde{W}} f'^* g_! \\ {\scriptstyle \mathrm{bc}_!}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (1.12)} \\ \tilde{f}_s'^* \tilde{g}_{s!} \mathrm{sp}_{\tilde{Y}} & & \mathrm{sp}_{\tilde{W}} g'_! f^* \\ {\scriptstyle (1.12)}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \mathrm{bc}_!} \\ \tilde{g}'_{s!} \tilde{f}_s^* \mathrm{sp}_{\tilde{Y}} & \xrightarrow[\mathrm{bc}^*]{} & \tilde{g}'_{s!} \mathrm{sp}_{\tilde{X}} f^* \end{array}$$

Indeed, by 1, this is equivalent to the commutativity of

$$\begin{array}{ccc} \tilde{f}_s'^* \mathrm{sp}_{\tilde{Z}} g_* & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{\tilde{W}} f'^* g_* \\ {\scriptstyle \mathrm{bc}_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (1.12)} \\ \tilde{f}_s'^* \tilde{g}_{s*} \mathrm{sp}_{\tilde{Y}} & & \mathrm{sp}_{\tilde{W}} g'_* f^* \\ {\scriptstyle (1.12)}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{bc}_*} \\ \tilde{g}'_{s*} \tilde{f}_s^* \mathrm{sp}_{\tilde{Y}} & \xrightarrow[\mathrm{bc}^*]{} & \tilde{g}'_{s*} \mathrm{sp}_{\tilde{X}} f^* \end{array}$$

and using the alternative description of bc_* this becomes a diagram with only one type of base change morphism, namely bc^*. It is then a straightforward exercise in decomposing diagrams to reduce the commutativity of it to the commutativity of pentagons as in 2.

A similar diagram commutes if $\tilde{f}$ and $\tilde{f}'$ are étale, with $\mathrm{bc}^!$ and bc_* replacing bc^* and $\mathrm{bc}_!$, respectively.

Now, we can use these constructions to define the specialization of a cohomological correspondence. Notice first that if X_1, X_2 are schemes over k and if we are given lifts $(\tilde{X}_i, \varphi_i)$, $i = 1, 2$, then there is a canonical lift $\tilde{X}_1 \times_R \tilde{X}_2$ of $X_1 \times X_2$. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
\tilde{X}_{1\eta} \times_\eta \tilde{X}_{2\eta} & \xrightarrow[\text{can}]{\cong} & \big(\tilde{X}_1 \times_R \tilde{X}_2\big)_\eta \\
{\scriptstyle \varphi_{1\eta} \times_\eta \varphi_{2\eta}} \downarrow & & \downarrow {\scriptstyle (\varphi_1 \times_R \varphi_2)_\eta} \\
X_{1\eta} \times_\eta X_{2\eta} & \xrightarrow[\text{can}]{\cong} & (X_{1R} \times_R X_{2R})_\eta
\end{array}$$

and the fact that the left vertical arrow is an isomorphism by assumption.

Suppose we are given a correspondence $c : C \to X_1 \times X_2$ over k and lifts $(\tilde{C}, \varphi_C)$, $(\tilde{X}_i, \varphi_i)$ of C, X_i, respectively. We can now say what it means to lift c to a correspondence $\tilde{c} : \tilde{C} \to \tilde{X}_1 \times_R \tilde{X}_2$ over R. By definition (considering c simply as a morphism in **Sch**$/k$), the diagram

$$\begin{array}{ccccc}
C_R & \xrightarrow{c_R} & (X_1 \times X_2)_R & \xrightarrow{\cong} & X_{1R} \times_R X_{2R} \\
{\scriptstyle \varphi_C} \uparrow & & & & \uparrow {\scriptstyle \varphi_1 \times_R \varphi_2} \\
\tilde{C} & & \xrightarrow[\tilde{c}]{} & & \tilde{X}_1 \times_R \tilde{X}_2
\end{array}$$

should be commutative. But this is equivalent to $(\varphi_C, \varphi_1, \varphi_2)$ being a morphism of correspondences over R. Thus we could have defined a lift of a correspondence c over k to be a correspondence $\tilde{c}$ over R together with a morphism of correspondences $\tilde{c} \to c_R$ over R which is an isomorphism over the generic fiber.

Definition 2.20 Let $c : C \to X_1 \times X_2$ be a correspondence over k, $\tilde{c} : \tilde{C} \to \tilde{X}_1 \times_R \tilde{X}_2$ a correspondence lifting c. For any $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$, we define the specialization map

$$\mathrm{sp}_{\tilde{c}} : \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Hom}_{\tilde{c}_s}(\mathrm{sp}_{\tilde{X}_1}\mathscr{F}_1, \mathrm{sp}_{\tilde{X}_2}\mathscr{F}_2)$$

as follows. Denote, as usual, by $\tilde{c}_s$ the fiber of $\tilde{c}$ over s. Given $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$ let $\mathrm{sp}_{\tilde{c}}(u)$ be the composition

$$\tilde{c}_{s2!}\tilde{c}_{s1}^*\mathrm{sp}_{\tilde{X}_1}\mathscr{F}_1 \xrightarrow{\mathrm{bc}^*} \tilde{c}_{s2!}\mathrm{sp}_{\tilde{C}}c_1^*\mathscr{F}_1 \xrightarrow{\mathrm{bc}_!} \mathrm{sp}_{\tilde{X}_2}c_{2!}c_1^*\mathscr{F}_1 \xrightarrow{\mathrm{sp}_{\tilde{X}_2}u} \mathrm{sp}_{\tilde{X}_2}\mathscr{F}_2.$$

$\mathrm{sp}_{\tilde{c}}(u)$ is called the *specialization of u (with respect to $\tilde{c}$)*.

Specialization thus provides a means to connect a correspondence c to the correspondence $\tilde{c}_s$ via the specialization map on cohomological correspondences lifting c. It is therefore quite natural to view a lift $\tilde{c}$ as a *morphism* from c to $\tilde{c}_s$. Under this view, the trivial lift c_R clearly corresponds to the identity morphism. What one gets is then the structure of a digraph, i. e. a category without composition. We denote the digraph one gets in this way starting from self-correspondences alone by **SCor**. We will take up this view in the next section.

Lemma 2.21 *Let X be a scheme over k, and let $\tilde{X} = X_R$ be the trivial lift. Denote, as in 1.(0), by $j : X_{\overline{\eta^h}} \to X_{\overline{R^h}}$ and $i : X = X_s \to X_{\overline{R^h}}$ the canonical morphisms. Then, for every $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$, the map*

$$\mathscr{F} \xrightarrow{\cong} i^*\mathscr{F}_{\overline{R^h}} \xrightarrow{\mathrm{adj}} i^*j_*j^*\mathscr{F}_{\overline{R^h}} \xrightarrow{\cong} \mathrm{sp}_{\tilde{X}}\mathscr{F} \qquad (2.10)$$

is an isomorphism.

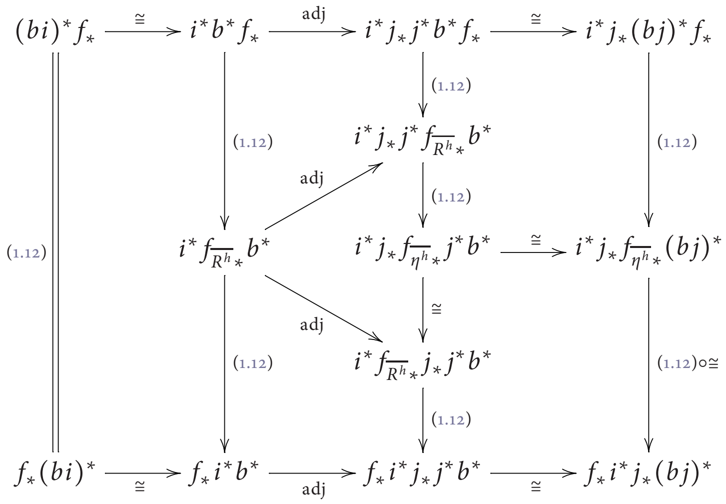
The proof of this lemma will be given in paragraph §7. Here, we can however prove its compatibility with respect to the base change morphisms introduced above.

Lemma 2.22 *Let $f : X \to Y$ be a morphism of schemes and let $\tilde{f} : X_R \to Y_R$ be the trivial lift of f. Then all the following four squares commute:*

$$\begin{array}{ccc} f_* & \xrightarrow{(2.10)} & \mathrm{sp}_{Y_R} f_* \\ \| & & \downarrow \mathrm{bc}_* \\ f_* & \xrightarrow{(2.10)} & f_* \mathrm{sp}_{X_R} \end{array} \qquad \begin{array}{ccc} f_! & \xrightarrow{(2.10)} & \mathrm{sp}_{Y_R} f_! \\ \| & & \uparrow \mathrm{bc}_! \\ f_! & \xrightarrow{(2.10)} & f_! \mathrm{sp}_{X_R} \end{array}$$

$$\begin{array}{ccc} f^* & \xrightarrow{(2.10)} & \mathrm{sp}_{X_R} f^* \\ \| & & \uparrow \mathrm{bc}^* \\ f^* & \xrightarrow{(2.10)} & f^* \mathrm{sp}_{Y_R} \end{array} \qquad \begin{array}{ccc} f^! & \xrightarrow{(2.10)} & \mathrm{sp}_{X_R} f^! \\ \| & & \downarrow \mathrm{bc}^! \\ f^! & \xrightarrow{(2.10)} & f^! \mathrm{sp}_{Y_R} \end{array}$$

Proof The proof is essentially the same in each of the four cases and we will do only one. Denoting the morphisms $X_{\overline{R^h}} \to X$ and $Y_{\overline{R^h}} \to Y$ by b (so that $bi = \mathbb{1}$), the first diagram may be expanded as follows:



The left half of this diagram commutes by the compatibility of (1.12) with respect to composition, as does the right upper square. The triangle commutes by the definition of (1.12) (cf. [2, XII, 4]) and the remaining two trapezoids and the lower right square clearly commute hence so does the whole diagram. □

Notation 2.23 Let X be a scheme over k and $\tilde{X}$ a lift of X over R. Also, let $\pi : X \to k$ and $\tilde{\pi} : \tilde{X} \to R$ be the structure morphisms. Then the composition

$$\pi_* \pi^! \Lambda_k \xrightarrow[\cong]{(2.10)} \mathrm{sp}_R \pi_* \pi^! \Lambda_k \xrightarrow{\mathrm{bc}_*} \tilde{\pi}_{s*} \mathrm{sp}_{\tilde{X}} \pi^! \Lambda_k \xrightarrow{\mathrm{bc}^!} \tilde{\pi}_{s*} \tilde{\pi}_s^! \mathrm{sp}_R \Lambda_k \xrightarrow[\cong]{(2.10)^{-1}} \tilde{\pi}_{s*} \tilde{\pi}_s^! \Lambda_k$$

induces a map $\mathrm{H}^0(X, K_X) \to \mathrm{H}^0(\tilde{X}_s, K_{\tilde{X}_s})$ which we denote by $\mathrm{sp}_{\tilde{X}}$.

Lemma 2.24 *Let X be a scheme over k and let $\tilde{X} = X_R$ be the trivial lift. Then the map*

$$\mathrm{sp}_{\tilde{X}} : \mathrm{H}^0(X, K_X) \longrightarrow \mathrm{H}^0(\tilde{X}_s, K_{\tilde{X}_s}) = \mathrm{H}^0(X, K_X)$$

is the identity.

Proof Consider the following diagram:

$$\begin{array}{ccc}
\pi_*\pi^!\Lambda_k & \xrightarrow{(2.10)} & \mathrm{sp}_R\pi_*\pi^!\Lambda_k \\
\| & & \downarrow{\scriptstyle \mathrm{bc}_*} \\
\pi_*\pi^!\Lambda_k & \xrightarrow{(2.10)} & \pi_*\mathrm{sp}_{X_R}\pi^!\Lambda_k \\
\| & & \downarrow{\scriptstyle \mathrm{bc}^!} \\
\pi_*\pi^!\Lambda_k & \xrightarrow{(2.10)} & \pi_*\pi^!\mathrm{sp}_R\Lambda_k
\end{array}$$

Both squares commute by 2.22 thus the claim. □

We end this paragraph with a result on lifting compactifications of correspondences. First we need a lemma.

Lemma 2.25 *Let X be a scheme over k and $(\tilde{X}, \varphi)$ a lift of X over R. Then there exists a scheme $\tilde{Y}$ over R such that*

1. *$\tilde{Y}$ is a lift of X over R,*
2. *$\tilde{Y}$ is proper over $\tilde{X}$, and*
3. *for every compactification $\overline{X}$ of X, $\tilde{Y}$ embeds as an open subscheme into a scheme $\tilde{Y}(\overline{X})$ proper over R and lifting $\overline{X}$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\tilde{Y} & \hookrightarrow & \tilde{Y}(\overline{X}) \\
\downarrow & & \\
\tilde{X} & & \downarrow \\
\downarrow & & \\
X_R & \hookrightarrow & \overline{X}_R
\end{array} \tag{2.11}$$

Proof We first reduce to the case where $\tilde{X}$ is proper over X_R. For this, let $j : \tilde{X} \hookrightarrow \tilde{X}''$ be a compactification over R and set $\tilde{X}'$ the scheme-theoretic closure of the image of the open immersion

$$(j, \varphi) : \tilde{X} \hookrightarrow \tilde{X}'' \times_R X_R.$$

Over η, we get a factorization

$$\tilde{X}_\eta \hookrightarrow \tilde{X}'_\eta \longrightarrow X_\eta,$$

where the first map is a dominant open immersion and the second proper. But since $\tilde{X}$ is a lift of X the composition is an isomorphism thus proper, hence the first map is proper thus surjective. Then it must be an isomorphism and consequently so is the second map. This shows that $\tilde{X}'$ is a lift of X as well. In addition it is proper over X_R by construction.

Suppose we can prove the lemma for $\tilde{X}'$, i. e. we can prove the existence of $\tilde{Y}'$ satisfying 1–3 (with $\tilde{X}'$ and $\tilde{Y}'$ replacing $\tilde{X}$ and $\tilde{Y}$, respectively). We claim that the fiber product $\tilde{Y} := \tilde{Y}' \times_{\tilde{X}'} \tilde{X}$ then satisfies 1–3 (for $\tilde{X}$). Indeed, $\tilde{X}_\eta \to \tilde{X}'_\eta$ is an isomorphism as we have shown above, which implies that its base change $\tilde{Y}_\eta \to \tilde{Y}'_\eta$ is an isomorphism as well. Since $\tilde{Y}'$ is a lift of X this implies 1. 2 holds since $\tilde{Y} \to \tilde{X}$ is a base change of the proper $\tilde{Y}' \to \tilde{X}'$, and finally for 3, the same scheme $\tilde{Y}'(\overline{X})$ which works in the case of $\tilde{X}'$ and $\tilde{Y}'$ works here too.

Thus we may assume that $\tilde{X}$ is proper over X_R. Let Z be the scheme-theoretic closure of $\tilde{X}_\eta$ in $\tilde{X}$ and denote by $f : Z \to X_R$ the induced proper morphism. Applying the next lemma below with $S = X_R$ and $U = X_\eta \subset X_R$ yields blow-ups Z', S' and an isomorphism $Z' \to S'$ as in the lemma. Set $\tilde{Y} = S'$. By the choice of the center of the blow-up $\tilde{Y}$ is a lift of X, i. e. 1 is satisfied. Moreover, the composition

$$\tilde{Y} \xrightarrow{\cong} Z' \longrightarrow Z \hookrightarrow \tilde{X}$$

of proper maps proves 2. For 3, let $X \hookrightarrow \overline{X}$ be a compactification, and denote by $V \subset X_R$ the center of the blow-up $\tilde{Y} \to X_R$. Let $\overline{V}$ be the scheme-theoretic closure of V in $\overline{X}_R$ and set $\tilde{Y}(\overline{X})$ to be the blow-up of $\overline{X}_R$ along $\overline{V}$. Clearly, $\tilde{Y}(\overline{X})$ is proper over R. Moreover, the center $\overline{V}$ does not meet $\overline{X}_\eta$ hence $\tilde{Y}(\overline{X})$ is a lift of $\overline{X}$. Finally, $\tilde{Y}$ embeds as an open subscheme into $\tilde{Y}(\overline{X})$, which proves 3. □

Lemma 2.26 *Let $f : Z \to S$ be a proper morphism of schemes, and let $U \subset S$ be a dense open subset such that $f^{-1}(U) \subset Z$ is dense and $f^{-1}(U) \to U$ an isomorphism. Then there exist blow-ups $Z' \to Z$ and $S' \to S$ with center away from $f^{-1}(U)$ and U, respectively, and an isomorphism $Z' \to S'$ rendering the following diagram commutative:*

$$\begin{array}{ccc} Z' & \xrightarrow{\cong} & S' \\ \downarrow & & \downarrow \\ Z & \xrightarrow[f]{} & S \end{array}$$

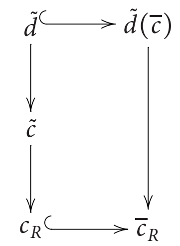
Proof This is proved (with additional hypotheses which are automatically satisfied in our noetherian setting) in [5, Theorem 2.11]. □

In the following corollary, all correspondences are assumed to be self-correspondences although, as is clear from the proof, it holds also in the more general context.

Corollary 2.27 *Let c be a correspondence and let $\tilde{c}$ be a correspondence lifting c. Then there exists a correspondence $\tilde{d}$ over R such that*

1. *$\tilde{d}$ lifts c,*
2. *$\tilde{d}$ is proper over $\tilde{c}$, and*
3. *for every compactification $\overline{c}$ of c, $\tilde{d}$ embeds with an open immersion into a correspondence $\tilde{d}(\overline{c})$*

proper over R and lifting $\overline{c}$ such that the following diagram commutes:

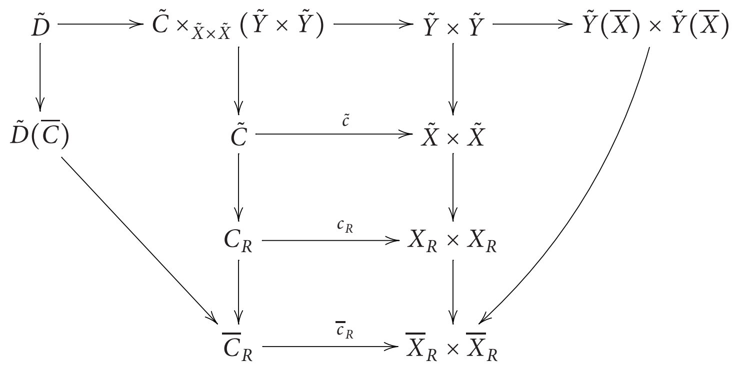


(2.12)

Proof Let us fix the following notation: $c : C \to X \times X$, $\tilde{c} : \tilde{C} \to \tilde{X} \times \tilde{X}$. (In this proof we abstain from writing R as the index to the fiber product.) First choose a lift $\tilde{Y}$ of X proper over $\tilde{X}$ as in Lemma 2.25. Then the scheme $\tilde{C} \times_{\tilde{X}\times\tilde{X}} (\tilde{Y} \times \tilde{Y})$ is a lift of C hence we may choose another lift $\tilde{D}$ proper over $\tilde{C} \times_{\tilde{X}\times\tilde{X}} (\tilde{Y} \times \tilde{Y})$ again as in Lemma 2.25. We claim that the correspondence

$$\tilde{d} : \tilde{D} \to \tilde{C} \times_{\tilde{X}\times\tilde{X}} (\tilde{Y} \times \tilde{Y}) \to \tilde{Y} \times \tilde{Y}$$

satisfies the conclusion of the corollary. Indeed, 1 and 2 follow immediately from the definition of $\tilde{d}$. For 3, let $\overline{c} : \overline{C} \to \overline{X} \times \overline{X}$ be any compactification of c. Choose $\tilde{Y}(\overline{X})$ and $\tilde{D}(\overline{C})$ as in Lemma 2.25. Then consider the following diagram:



The trapezoid on the left as well as the triangle on the right commute by Lemma 2.25: they are instances of (2.11). The bottom square in the middle is a base change of a commutative diagram and thus commutative, the middle square commutes because $\tilde{c}$ lifts c, while the top square clearly commutes. Hence the whole diagram is commutative and we deduce an open immersion

$$\tilde{D} \hookrightarrow \tilde{D}(\overline{C}) \times_{\overline{X}_R \times \overline{X}_R} (\tilde{Y}(\overline{X}) \times \tilde{Y}(\overline{X})).$$

Clearly, the target of this morphism is proper over R and lifts $\overline{C}$. We deduce from the commutativity of the diagram above that the correspondence

$$\tilde{d}(\overline{c}) : \tilde{D}(\overline{C}) \times_{\overline{X}_R \times \overline{X}_R} (\tilde{Y}(\overline{X}) \times \tilde{Y}(\overline{X})) \to \tilde{Y}(\overline{X}) \times \tilde{Y}(\overline{X})$$

satisfies the conclusion of part 3 of the corollary. □

§6 Deformation to the normal cone

Let Z be a closed subscheme of a scheme X. Recall that the deformation to the normal cone associated to the pair (X, Z) usually refers to the construction of a scheme $\tilde{X}_Z$ and a commutative diagram

$$\begin{array}{ccc} Z \times \mathbb{P}^1 & \hookrightarrow & \tilde{X}_Z \\ & \searrow^{\mathrm{pr}_2} & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

such that over $\mathbb{A}^1 = \mathbb{P}^1 \backslash \{\infty\}$ the embedding is the canonical embedding $Z \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$ while over ∞ it can be identified with the zero-section embedding of Z into the normal cone to Z in X (cf. [10, chapter 5]). We will now explain this construction more precisely when $\mathbb{P}^1$ is replaced by a discrete valuation ring R as in the previous paragraph, $\mathbb{A}^1$ by the generic fiber η and ∞ by the closed fiber s. This will allow us to apply the specialization formalism introduced above.

Throughout this paragraph we fix a discrete valuation ring R as in the previous paragraph. Denote by t any uniformizer of R, and consider, for $n \geq 0$, the R-submodules $Rt^{-n} \subset R[1/t] = K$ of the fraction field of R. Pulling back the associated quasi-coherent sheaves of $\mathscr{O}_R$-modules along the projection map $p_R : X_R \to R$ we get quasi-coherent sheaves of $\mathscr{O}_{X_R}$-modules

$$\mathscr{O}_{X_R} t^{-n} := p_R^{\circledast} \widetilde{Rt^{-n}} = p_R^* \widetilde{Rt^{-n}} \otimes_{p_R^* \mathscr{O}_R} \mathscr{O}_{X_R} \subset p_R^* \widetilde{R[1/t]} \otimes_{p_R^* \mathscr{O}_R} \mathscr{O}_{X_R} = p_R^{\circledast} \widetilde{R[1/t]} =: \mathscr{O}_{X_R}[1/t].$$

Let $\mathscr{I}_{Z_R} \subset \mathscr{O}_{X_R}$ be the ideal sheaf corresponding to Z_R. We may define the following sheaf of $\mathscr{O}_{X_R}$-algebras:

$$\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] := \sum_{n \geq 0} \mathscr{O}_{X_R} t^{-n} \otimes_{\mathscr{O}_{X_R}} \mathscr{I}_{Z_R}^n, \tag{2.13}$$

where $\mathscr{I}_{Z_R}^0 = \mathscr{O}_{X_R}$ and where its algebra structure comes from the natural embedding $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \hookrightarrow \mathscr{O}_{X_R}[1/t]$. Notice that all the summands in (2.13) are quasi-coherent therefore so is $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$. Locally, e. g. when $X = \mathrm{Spec}(A)$ and Z corresponds to an ideal $I \subset A$, $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ corresponds to the $A \otimes_k R$-algebra

$$\sum_{n \geq 0} (I \otimes_k R)^n \otimes_R Rt^{-n} \cong \sum_{n \geq 0} I^n \otimes_k Rt^{-n} \subset A \otimes_k R[1/t]. \tag{2.14}$$

An alternative description of $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ can be given as follows. Denote by u the canonical open immersion $X_\eta \hookrightarrow X_R$. Then we have an isomorphism of $\mathscr{O}_{X_R}$-algebras $\mathscr{O}_{X_R}[1/t] \cong u_* \mathscr{O}_{X_\eta}$, and $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ is the $\mathscr{O}_{X_R}$-subalgebra generated by $u^\sharp(\mathscr{I}_{Z_R}) \cdot 1/t \subset u_* \mathscr{O}_{X_\eta}$.

Thus we may associate to any X, Z and t as above, a scheme $\tilde{X}_Z = \mathbf{Spec}(\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t])$ together with an affine morphism $\tilde{\varphi} = \tilde{\varphi}_{X,Z} : \tilde{X}_Z \to X_R$ corresponding to the inclusion $\varphi : \mathscr{O}_{X_R} \hookrightarrow \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ (cf. [11, §1]). Some of the properties of this construction, analogous to the deformation to the normal cone described above, are given in the following lemma.

Lemma 2.28 *Let X, Z, and t be as above. Then:*

1. *The isomorphism class of $\tilde{X}_Z$ does not depend on the choice of t.*
2. *The pair $(\tilde{X}_Z, \tilde{\varphi})$ is a lift of X.*
3. *The fiber of $\tilde{X}_Z$ over s is isomorphic to the normal cone $N_Z(X)$ to Z in X.*
4. *There is a canonical closed embedding $\tilde{i} : Z_R \hookrightarrow \tilde{X}_Z$ lifting $i : Z \hookrightarrow X$ which, over s, identifies Z with the zero section of $N_Z(X)$.*

5. *The two objects* $(\tilde{X}_Z)_{red}$ *and* $\widetilde{(X_{red})}_{Z_{red}}$ *are canonically identified as schemes over* X_R.

Proof 1. This is obvious.

2. First of all, $\tilde{X}_Z$ is separated and of finite type over R. Indeed, X_R has these properties and $\tilde{X}_Z$ is affine thus separated over X_R. Moreover, $\tilde{X}_Z$ being of finite type over X_R is equivalent to $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ being a finite-type $\mathscr{O}_{X_R}$-algebra ([11, 1.3.7]). But this is true since $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ is generated by $\mathscr{O}_{X_R} t^{-1} \otimes_{\mathscr{O}_{X_R}} \mathscr{I}^1_{Z_R}$.

   Next, we need to show that the left vertical arrow in the following cartesian square is an isomorphism:

$$\begin{array}{ccc} (\tilde{X}_Z)_\eta & \longrightarrow & \tilde{X}_Z \\ {\scriptstyle \tilde{\varphi}_\eta}\downarrow & & \downarrow{\scriptstyle \tilde{\varphi}} \\ X_\eta & \xrightarrow[\tilde{u}]{} & X_R \end{array}$$

   $\tilde{\varphi}_\eta$ corresponds to the morphism

$$\mathscr{O}_{X_R} \otimes_{\mathscr{O}_{X_R}} \tilde{u}_* \mathscr{O}_{X_\eta} \xrightarrow{\varphi \otimes \mathbb{1}} \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \otimes_{\mathscr{O}_{X_R}} \tilde{u}_* \mathscr{O}_{X_\eta} \tag{2.15}$$

   (cf. [11, 1.4.6]) and it thus suffices to show that (2.15) is an isomorphism ([11, 1.2.8]). For this we may assume $X = \operatorname{Spec}(A)$, Z corresponding to an ideal $I \subset A$. Using (2.14) we see that (2.15) can be identified with the canonical inclusion

$$A \otimes_k R[1/t] \longrightarrow \sum_{n \geq 0} I^n \otimes_k R[1/t] = A \otimes_k R[1/t].$$

3. The quasi-coherent $\mathscr{O}_{X_R}$-algebra corresponding to the scheme $(\tilde{X}_Z)_s$ over X_R is

$$\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \otimes_{\mathscr{O}_{X_R}} \tilde{v}_* \mathscr{O}_{X_s},$$

   where $\tilde{v} : X_s \hookrightarrow X_R$ is the canonical closed embedding. Since $N_Z(X) = \mathbf{Spec}(\oplus_{n \geq 0} \mathscr{I}^n_Z / \mathscr{I}^{n+1}_Z)$ it thus suffices to give an isomorphism

$$\oplus_{n \geq 0} \mathscr{I}^n_Z / \mathscr{I}^{n+1}_Z \longrightarrow \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \otimes_{\mathscr{O}_{X_R}} \tilde{v}_* \mathscr{O}_{X_s}$$

   of $\mathscr{O}_{X_R}$-algebras. Locally, i. e. when $X = \operatorname{Spec}(A)$, Z corresponding to an ideal $I \subset A$, the right hand side is

$$\left(\sum_{n \geq 0} I^n \otimes_k R t^{-n} \right) \otimes_R R/tR \cong \frac{\sum_{n \geq 0} I^n \otimes_k R t^{-n}}{\sum_{n \geq 0} I^n \otimes_k R t^{-(n-1)}}$$

   and we may define the map by $([a_n])_{n \geq 0} \mapsto \sum_{n \geq 0} [a_n \otimes t^{-n}]$, where $a_n \in I^n$. This is clearly well-defined. The inverse map is given by $\sum_{n \geq 0} [a_n \otimes r_n t^{-n}] \mapsto ([\overline{r_n} a_n])_{n \geq 0}$ where $a_n \in I^n$, $r_n \in R$ and where $\overline{r_n}$ denotes the image of r_n under the projection $R \to R/tR = k$. This is also easily seen to be well-defined. Finally, these isomorphisms extend to the non-affine case which is easy to see because the intersection of two open affines is again an open affine.

4. The canonical surjective projection $p : \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \to \mathscr{O}_{X_R}/\mathscr{I}_{Z_R} = \mathscr{O}_{Z_R}$ induces a closed immersion $Z_R \hookrightarrow \tilde{X}_Z$ ([11, 1.4.10]) which, over s, corresponds to the morphism

$$\oplus_{n \geq 0} \mathscr{I}^n_Z / \mathscr{I}^{n+1}_Z \xrightarrow{\cong} \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \otimes_{\mathscr{O}_{X_R}} \tilde{v}_* \mathscr{O}_{X_s} \xrightarrow{p \otimes \mathbb{1}} \mathscr{O}_{X_R}/\mathscr{I}_{Z_R} \otimes_{\mathscr{O}_{X_R}} \tilde{v}_* \mathscr{O}_{X_s}.$$

Locally, when $X = \mathrm{Spec}(A)$, Z corresponding to an ideal $I \subset A$, this is the morphism

$$\oplus_{n\geq 0} I^n/I^{n+1} \longrightarrow \frac{\sum_{n\geq 0} I^n \otimes_k Rt^{-n}}{\sum_{n\geq 0} I^n \otimes_k Rt^{-(n-1)}} \longrightarrow A/I$$

which maps $([a_n])_{n\geq 0}$ to $[a_0]$, thus the identification of Z with the zero section. The claim about the so defined $\tilde{i}$ lifting i amounts to saying that the composition

$$\mathcal{O}_{X_R} \overset{\varphi}{\hookrightarrow} \mathcal{O}_{X_R}[\mathcal{I}_{Z_R}/t] \overset{p}{\longrightarrow} \mathcal{O}_{X_R}/\mathcal{I}_{Z_R}$$

corresponds to the inclusion of Z_R into X_R, which is obvious.

5. Again, we will give the identification only locally. Thus assume $X = \mathrm{Spec}(A)$, Z corresponding to the ideal I. Consider the following map:

$$\begin{aligned} \sum_{n\geq 0} I^n \otimes_k Rt^{-n} &\longrightarrow \sum_{n\geq 0} (\sqrt{I}^n + \sqrt{0})/\sqrt{0} \otimes_k Rt^{-n} \\ a_n \otimes r_n t^{-n} &\longmapsto [a_n] \otimes r_n t^{-n}. \end{aligned}$$

This is clearly a well-defined surjective $A \otimes_k R$-algebra morphism. To determine its kernel suppose $\sum_{i=1}^m [a_{n_i}] \otimes r_i t^{-n_i} = 0$ in $A \otimes_k R[1/t]$, where $a_{n_i} \in I^{n_i}$, $r_i \in R\backslash(t)$, no two n_i's equal. Then the $r_i t^{-n_i}$'s are k-linearly independent in $R[1/t]$ which implies that $[a_{n_i}] = 0$ for all i. Let $k_i \geq 0$ such that $a_{n_i}^{k_i} = 0$ and set $k_0 = (\max_{i=1,\dots,m}\{k_i\} - 1) \cdot m + 1$. For every k_0-tuple $(i_1, \dots, i_{k_0}) \in \{1, \dots, m\}^{k_0}$ we then have $\prod_{j=1}^{k_0} a_{n_{i_j}} = 0$ in A which implies that

$$(\sum_{i=1}^m a_{n_i} \otimes r_i t^{-n_i})^{k_0} = 0$$

in A, i. e. the kernel of our map is contained in the nilradical of A. Conversely, the image of the map is a reduced ring ([12, 4.6.1]), hence the nilradical of A is contained in the kernel. This yields the isomorphism we were looking for. □

Notation 2.29 To investigate the functorial properties of the above construction we introduce the ad-hoc category **ahc** whose objects are pairs of schemes (X, Z) over k, $Z \subset X$ a closed subscheme, and whose morphisms $f : (X, Z) \to (X', Z')$ are morphisms of schemes $f : X \to X'$ such that Z is a closed subscheme of $f^{-1}(Z')$; composition of morphisms and identity morphisms are canonical. Such a morphism f is called *special* if in fact $f^{-1}(Z') = Z$.

Lemma 2.30 *Let $f : (X, Z) \to (X', Z')$ be a morphism in* **ahc**.

1. *There is a unique lift $\tilde{f} : \tilde{X}_Z \to \tilde{X}'_{Z'}$ of f such that the following diagram commutes:*

$$\begin{array}{ccc} X_\eta & \overset{f_\eta}{\longrightarrow} & X'_\eta \\ \downarrow & & \downarrow \\ \tilde{X}_Z & \underset{\tilde{f}}{\longrightarrow} & \tilde{X}'_{Z'} \end{array} \qquad (2.16)$$

Moreover, the association $(X, Z) \mapsto \tilde{X}_Z$, $f \mapsto \tilde{f}$, defines a covariant functor from **ahc** *to* **Sch**$/R$.

2. *The image $\tilde{f}_s(N_Z(X))$ is contained set-theoretically in the zero-section $Z' \subset N_{Z'}(X')$ if and only if there exists $k_0 \in \mathbb{N}$ such that $f^*(\mathscr{I}_{Z'}^{k_0}) \cdot \mathscr{O}_X \subset \mathscr{I}_Z^{k_0+1}$.*

3. *$\tilde{f}_s|_Z : Z \to Z'$ is equal to $f|_Z$.*

4. *If f is special then the map $(\tilde{f}, \tilde{\varphi}) : \tilde{X}_Z \to \tilde{X}'_{Z'} \times_{X'_R} X_R$ is a closed immersion. It is an open immersion if f is in addition flat.*

Proof 1. To give a lift of f, $\tilde{f} : \tilde{X}_Z \to \tilde{X}'_{Z'}$, is equivalent to give a morphism of $f_R^* \mathscr{O}_{X'_R}$-algebras $f_R^* \mathscr{O}_{X'_R}[\mathscr{I}_{Z'_R}/t] \to \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ ([11, 1.5.6]), i. e. to make the lower half of the following diagram commutative:

$$\begin{array}{ccc}
f_R^* \mathscr{O}_{X'_R}[1/t] & \xrightarrow{f_\eta^\flat} & \mathscr{O}_{X_R}[1/t] \\
\uparrow & & \uparrow \\
f_R^* \mathscr{O}_{X'_R}[\mathscr{I}_{Z'_R}/t] & \dashrightarrow & \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t] \\
\uparrow & & \uparrow \\
f_R^* \mathscr{O}_{X'_R} & \xrightarrow[f_R^\sharp]{} & \mathscr{O}_{X_R}
\end{array}$$

(Here, the upper horizontal arrow corresponds to f_η.) On the other hand, commutativity of (2.16) means that the upper half of this diagram commutes thus if there is at least one lift as specified in the lemma then it is clearly unique.

It remains to check that the generators of $f_R^* \mathscr{O}_{X'_R}[\mathscr{I}_{Z'_R}/t]$ as an $f_R^* \mathscr{O}_{X'_R}$-algebra are mapped into $\mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t]$ under $f_\eta^\flat$. For this notice that the morphism $f_R^* \mathscr{O}_{X'_R} \to f_R^* u'_* \mathscr{O}_{X'_\eta} \to u_* \mathscr{O}_{X_\eta}$ can be written as $u^\sharp \circ f_R^\sharp$, where $u : X_\eta \to X_R$, hence the image of the generators is contained in

$$u^\sharp \circ f_R^\sharp(f_R^* \mathscr{I}_{Z'_R}) \cdot 1/t \subset u^\sharp(\mathscr{I}_{Z_R}) \cdot 1/t \subset \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t],$$

by our assumption that Z is a closed subscheme of $f^{-1}(Z')$ hence that Z_R is a closed subscheme of $f^{-1}(Z')_R = f_R^{-1}(Z'_R)$.

That the association defines a functor follows immediately from the uniqueness statement proved.

2. First assume there is such a $k_0 \in \mathbb{N}$. To prove $\tilde{f}_s(N_Z(X)) \subset Z'$ set-theoretically we may assume $X' = \mathrm{Spec}(A')$, Z' corresponding to an ideal $I' \subset A'$, $X = \mathrm{Spec}(A)$, Z corresponding to an ideal $I \subset A$. In this case, $f_\eta^\sharp$ is simply $f^\sharp \otimes \mathbb{1} : A' \otimes_k R[1/t] \to A \otimes_k R[1/t]$ hence, using the isomorphisms constructed in 2.28.3, $\tilde{f}_s^\sharp$ can be described as follows:

$$\begin{array}{ccccccc}
\oplus_{n\geq 0} I'^n/I'^{n+1} & \ni & ([a_n])_{n\geq 0} & & ([f^\sharp(a_n)])_{n\geq 0} & \in & \oplus_{n\geq 0} I^n/I^{n+1} \\
& & \downarrow & & \uparrow & & \\
\frac{A'\otimes_k R[I'\otimes 1/t]}{tR\cdot(A'\otimes_k R[I'\otimes 1/t])} & \ni & \sum_{n\geq 0}[a_n \otimes t^{-n}] & \mapsto & \sum_{n\geq 0}[f^\sharp(a_n) \otimes t^{-n}] & \in & \frac{A\otimes_k R[I\otimes 1/t]}{tR\cdot(A\otimes_k R[I\otimes 1/t])}
\end{array}$$

Thus by assumption, for every $n \geq 1$ and $a_n \in I'^n$,

$$\tilde{f}_s^\sharp([a_n]_{I'^{n+1}})^{k_0} = \tilde{f}_s^\sharp([a_n^{k_0}]_{I'^{nk_0+1}}) = [f^\sharp(a_n^{k_0})]_{I^{nk_0+1}} = 0,$$

which implies that, for each prime ideal $p \in N_Z(X)$, $\oplus_{n\geq 1} I'^n/I'^{n+1} \subset (\tilde{f}_s^\sharp)^{-1}(p)$, i. e. $\tilde{f}_s(p) \in Z'$ (cf. 2.28.4).

Conversely, assume $\tilde{f}_s(N_Z(X)) \subset Z'$ set-theoretically. $f^\sharp(f^*\mathscr{I}_{Z'}^{k_0}) \subset \mathscr{I}_Z^{k_0+1}$ being a local property on both X and X', and X being quasi-compact, it suffices to find such a k_0 locally, i. e. when $X = \mathrm{Spec}(A)$, Z corresponding to an ideal $I \subset A$, $X' = \mathrm{Spec}(A')$, Z' corresponding to an ideal $I' \subset A'$. We may then use the description of $\tilde{f}_s^\sharp$ worked out above: $([a_n])_{n\geq 0} \mapsto ([f^\sharp(a_n)])_{n\geq 0}$. By assumption, $\tilde{f}^\sharp(\oplus_{n\geq 1} I'^n/I'^{n+1})$ is contained in every prime ideal of $N_Z(X)$ hence also in the radical $\sqrt{\oplus_{n\geq 0} I^n/I^{n+1}}$. Choose generators $r_1, \dots, r_m$ of I' and choose $k_i \in \mathbb{N}_{>0}$ such that $\tilde{f}^\sharp([r_i]_{I'^2})^{k_i} = 0$, $i = 1, \dots, m$. Set $k_0 = (\max_{i=1,\dots,m}\{k_i\} - 1) \cdot m + 1$. For every k_0-tuple $(i_1, \dots, i_{k_0}) \in \{1, \dots, m\}^{k_0}$, we then have

$$0 = \prod_{j=1}^{k_0} \tilde{f}^\sharp([r_{i_j}]_{I'^2}) = \tilde{f}^\sharp(\prod_{j=1}^{k_0}[r_{i_j}]_{I'^2}) = \tilde{f}^\sharp([\prod_{j=1}^{k_0} r_{i_j}]_{I^{k_0+1}}) = [f^\sharp(\prod_{j=1}^{k_0} r_{i_j})]_{I^{k_0+1}},$$

i. e. $f^\sharp$ maps a set of generators of I'^{k_0} into I^{k_0+1} which proves the claim.

3. This statement is local on both X and X'. It thus follows from the explicit description of $\tilde{f}_s$ given in the proof of part 2 when X and X' are both affine.

4. The map in the statement of the lemma corresponds to the morphism of $\mathscr{O}_{X_R}$-algebras

$$\tilde{f}^\flat \otimes \tilde{\varphi}^\sharp : f_R^\circledast \mathscr{O}_{X'_R}[\mathscr{I}_{Z'_R}/t] \longrightarrow \mathscr{O}_{X_R}[\mathscr{I}_{Z_R}/t], \tag{2.17}$$

where $\tilde{f}^\flat$ is the restriction of $f_\eta^\flat$ as in 1. It suffices to prove this morphism surjective (resp. flat if f is flat—since a flat closed immersion is automatically an open immersion). For this we may assume $X = \mathrm{Spec}(A)$, Z corresponding to an ideal $I \subset A$, $X' = \mathrm{Spec}(A')$, Z' corresponding to an ideal $I' \subset A'$. In this case, as above, $f_\eta^\sharp$ is simply $f^\sharp \otimes \mathbb{1} : A' \otimes_k R[1/t] \to A \otimes_k R[1/t]$, and (2.17) is the $A \otimes_k R$-morphism

$$(f^\sharp \otimes \mathbb{1}) \otimes \tilde{\varphi}^\sharp : (A' \otimes_k R[I' \otimes 1/t]) \otimes_{A' \otimes_k R} A \otimes_k R \longrightarrow A \otimes_k R[I \otimes 1/t], \tag{2.18}$$

where $\tilde{\varphi}^\sharp : A \otimes_k R \to A \otimes_k R[I \otimes 1/t]$ is the canonical inclusion. This morphism factors canonically as follows:

$$\begin{array}{ccc} (A' \otimes_k R[I' \otimes 1/t]) \otimes_{A' \otimes_k R} A \otimes_k R & \xrightarrow{(2.18)} & A \otimes_k R[I \otimes 1/t] \ , \\ \downarrow \cong & \nearrow \cong & \\ (A' \otimes_{A'} A) \otimes_k R[(I' \otimes_{A'} A) \otimes 1/t] & & \end{array}$$

If f is special then the ideal generated by $f^\sharp(I')$ in A equals I, hence the second morphism in the factorization is surjective. If f is flat, then we have $I' \otimes_{A'} A \cong I$. □

Corollary 2.31 *Let $f : (X, Z) \to (X', Z')$ be a special morphism in* **ahc**. *Then:*

1. $\tilde{f}_s^{-1}(Z') = Z \subset N_Z(X)$.

2. *If f is a closed immersion (resp. proper, smooth) then so is $\tilde{f}$.*

Proof 1. Consider the commutative diagram

$$\begin{array}{ccccccc}
Z' & \hookrightarrow & N_{Z'}(X') & \hookrightarrow & \tilde{X}'_{Z'} & & \\
\uparrow & & \uparrow \tilde{f}_s & & \uparrow \tilde{f} & \nwarrow & \\
\tilde{f}_s^{-1}(Z') & \hookrightarrow & N_Z(X) & \hookrightarrow & \tilde{X}_Z & \xhookrightarrow[(\tilde{f},\tilde{\varphi})]{} & \tilde{X}'_{Z'} \times_{X'_R} X_R
\end{array}$$

where all horizontal arrows are closed immersions, the last one in the bottom row being so by the last lemma. We deduce from it a closed immersion $\tilde{f}_s^{-1}(Z')$ into $Z' \times_{\tilde{X}'_{Z'}} \tilde{X}'_{Z'} \times_{X'_R} X_R = Z' \times_{X'_R} X_R = Z$.

On the other hand, the maps $\tilde{f}_s|_Z : Z \to Z'$ and $Z \hookrightarrow N_Z(X)$ define a map $Z \to \tilde{f}_s^{-1}(Z')$ whose composition with the closed immersion $\tilde{f}_s^{-1}(Z') \hookrightarrow N_Z(X)$ is a closed immersion; thus it is a closed immersion itself. We have proved $Z = \tilde{f}_s^{-1}(Z')$.

2. Decompose $\tilde{f}$ as $\tilde{X}_Z \hookrightarrow \tilde{X}'_{Z'} \times_{X'_R} X_R \to \tilde{X}'_{Z'}$. If f is a closed immersion (resp. proper, smooth) then so is the second map. Taking into account part 4 of the last lemma, the claim follows. □

Definition 2.32 Let $c : C \to X \times X$ be a correspondence and $Z \subset X$ a closed subscheme. By 2.30.1, c lifts to a correspondence $\tilde{c}_Z : \tilde{C}_{c^{-1}(Z\times Z)} \to \tilde{X}_Z \times_R \tilde{X}_Z$ over R. We call it the *deformation of c (with respect to Z and R)*.

To state our main result in this particular situation which follows easily from what we just proved, let us introduce some notation.

Notation 2.33 Let $c : C \to X \times X$ be a correspondence. We define the *scheme of fixed points of c* to be the fiber product of $\Delta = \Delta_X$ and c,

$$\mathrm{Fix}(c) := X \times_{X\times X} C.$$

The closed immersion $\mathrm{Fix}(c) \hookrightarrow C$ will be denoted Δ' and the restriction of c to $\mathrm{Fix}(c)$ will be denoted $c' : \mathrm{Fix}(c) \to X$.

Corollary 2.34 *Let $c : C \to X\times X$ be a correspondence and let $Z \subset X$ be a closed subscheme. Then there is a unique closed immersion $\widetilde{\mathrm{Fix}(c)}_{c'^{-1}(Z)} \hookrightarrow \mathrm{Fix}(\tilde{c}_Z)$ making the following diagram commutative:*

$$\begin{array}{ccccc}
 & \swarrow^{\widetilde{\Delta'}} & \widetilde{\mathrm{Fix}(c)}_{c'^{-1}(Z)} & \searrow^{\widetilde{c'}} & \\
 & & \downarrow & & \\
\tilde{C}_{c^{-1}(Z\times Z)} & \xleftarrow[(\tilde{\Delta})']{} & \mathrm{Fix}(\tilde{c}_Z) & \xrightarrow[(\tilde{c}_Z)']{} & \tilde{X}_Z
\end{array} \tag{2.19}$$

Proof Notice first that the lift $\widetilde{\Delta'}$ in (2.19) really exists since

$$\Delta'^{-1}(c^{-1}(Z \times Z)) = c'^{-1}(\Delta^{-1}(Z \times Z)) = c'^{-1}(Z).$$

Moreover, $\widetilde{\Delta'}$ is, by 2.31.2, a closed immersion. By the functoriality proved in 2.30.1, we have

$$\tilde{\Delta} \circ \widetilde{c'} = \widetilde{\Delta \circ c'} = \widetilde{c \circ \Delta'} = \tilde{c}_Z \circ \widetilde{\Delta'},$$

hence the universal property of fiber products yields a unique morphism $\widetilde{\mathrm{Fix}(c)}_{c'^{-1}(Z)} \to \mathrm{Fix}(\tilde{c}_Z)$ making (2.19) commutative. Since its composition with the closed immersion $(\tilde{\Delta})'$ is a closed immersion, it has to be one itself. □

§7 Restriction to the summit

In this paragraph we will prove a property of the specialization functor analogous to (SP5) in [20], called "restriction au sommet" there: Let X, Z, R be as in the previous paragraph, i. e. X a scheme, $Z \subset X$ a closed subscheme and R a discrete valuation ring over k with residue field k. We have defined in 2.28.4 a canonical embedding $\tilde{i} : Z_R \hookrightarrow \tilde{X}_Z$ lifting $i : Z \hookrightarrow X$, to which thus corresponds a base change morphism $\mathrm{bc}^* : \tilde{i}_s^* \mathrm{sp}_{\tilde{X}_Z} \to \mathrm{sp}_{Z_R} i^*$ (cf. 2.§5).

Proposition 2.35 *For every $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ the morphism*

$$(\mathrm{sp}_{\tilde{X}_Z} \mathscr{F})|_Z \xrightarrow{\mathrm{bc}^*} \mathrm{sp}_{Z_R}(\mathscr{F}|_Z) \xrightarrow{(2.10)^{-1}} \mathscr{F}|_Z \tag{2.20}$$

is an isomorphism.

Remark 2.36 Notice that for this statement to make sense at all we have to show (2.10) an isomorphism. This will be done below and we will pay attention not to use any circular argument, i. e. such that the proof of 2.21 given below is independent of 2.35.

Set $U := X \backslash Z$ and denote by j the inclusion $U \hookrightarrow X$. Moreover let $C_{X,Z}(\mathscr{F})$ denote the cone of the morphism (2.20). Note the following useful fact.

Lemma 2.37 *Fix X and Z as above. Let $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to^+$ be a distinguished triangle in $\mathfrak{D}^b_{ctf}(X)$.*

1. *If $C_{X,Z}(\mathscr{F}_i) = C_{X,Z}(\mathscr{F}_j) = 0$, some $i, j \in \{1,2,3\}$, $i \neq j$, then $C_{X,Z}(\mathscr{F}_i) = 0$ for all $i \in \{1,2,3\}$.*
2. *Let $\xi \in Z$ be a point and assume $C_{X,Z}(\mathscr{F}_2)_{\overline{\xi}} = 0$. If $\mathrm{H}^{m-1}(C_{X,Z}(\mathscr{F}_3))_{\overline{\xi}}$ vanishes then so does $\mathrm{H}^m(C_{X,Z}(\mathscr{F}_1))_{\overline{\xi}}$.*

Proof
1. By rotating the triangle we may assume $\{i, j\} = \{2, 3\}$. Applying part 2 for all $m \in \mathbb{Z}$ and all points $\xi \in Z$, the claim follows.
2. From the distinguished triangle given we deduce a commutative diagram in $\mathfrak{D}^b_{ctf}(Z)$ (the dotted arrows exist by the axioms for a triangulated category):

$$\begin{array}{ccccccc}
(\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z & \xrightarrow{(2.20)} & \mathscr{F}_1|_Z & \longrightarrow & C_{X,Z}(\mathscr{F}_1) & \longrightarrow & (\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
(\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_2)|_Z & \xrightarrow{(2.20)} & \mathscr{F}_2|_Z & \longrightarrow & C_{X,Z}(\mathscr{F}_2) & \longrightarrow & (\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_2)|_Z[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
(\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_3)|_Z & \xrightarrow{(2.20)} & \mathscr{F}_3|_Z & \longrightarrow & C_{X,Z}(\mathscr{F}_3) & \longrightarrow & (\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_3)|_Z[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
(\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z[1] & \xrightarrow{(2.20)} & \mathscr{F}_1|_Z[1] & \longrightarrow & C_{X,Z}(\mathscr{F}_1)[1] & \longrightarrow & (\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z[2]
\end{array}$$

Notice that all rows, and all columns except possibly the third are distinguished triangles. Hence applying a suitable cohomology functor followed by the stalk functor $(-)_{\overline{\xi}}$ yields the following two commutative diagrams with exact columns:

$$\begin{array}{ccc}
\mathrm{H}^{m-1}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_3)|_Z)_{\overline{\xi}} & \twoheadrightarrow & \mathrm{H}^{m-1}(\mathscr{F}_3|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z)_{\overline{\xi}} & \longrightarrow & \mathrm{H}^{m}(\mathscr{F}_1|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_2)|_Z)_{\overline{\xi}} & \twoheadrightarrow & \mathrm{H}^{m}(\mathscr{F}_2|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_3)|_Z)_{\overline{\xi}} & \hookrightarrow & \mathrm{H}^{m}(\mathscr{F}_3|_Z)_{\overline{\xi}}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{H}^{m}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_2)|_Z)_{\overline{\xi}} & \twoheadrightarrow & \mathrm{H}^{m}(\mathscr{F}_2|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_3)|_Z)_{\overline{\xi}} & \hookrightarrow & \mathrm{H}^{m}(\mathscr{F}_3|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m+1}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_1)|_Z)_{\overline{\xi}} & \longrightarrow & \mathrm{H}^{m+1}(\mathscr{F}_1|_Z)_{\overline{\xi}} \\
\downarrow & & \downarrow \\
\mathrm{H}^{m+1}((\mathrm{sp}_{\tilde{X}_Z}\mathscr{F}_2)|_Z)_{\overline{\xi}} & \hookrightarrow & \mathrm{H}^{m+1}(\mathscr{F}_2|_Z)_{\overline{\xi}}
\end{array}$$

Using the four lemma (or an easy diagram chase) shows that the second row of the first diagram is surjective and the third row in the second diagram is injective. This implies the claim. □

We thus have to prove $C_{X,Z}(\mathscr{F}) = 0 \in \mathfrak{D}^b_{ctf}(Z)$ for any X,Z, and $\mathscr{F}$. We start with a result concerning the behavior of the specialization functor under base change.

Lemma 2.38 *Let $f : X \to X'$ be a morphism of schemes, let $Z' \subset X'$ be a closed subscheme and denote by $f_Z : Z \to Z'$ the restriction of f to $Z = f^{-1}(Z')$.*

1. *Let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and assume f is proper. Then we have*

$$C_{X',Z'}(f_!\mathscr{F}) \cong f_{Z!}C_{X,Z}(\mathscr{F}).$$

2. *Let $\mathscr{F}' \in \mathfrak{D}^b_{ctf}(X')$ and assume f is smooth. Then we have*

$$C_{X,Z}(f^*\mathscr{F}') \cong f_Z^* C_{X',Z'}(\mathscr{F}').$$

(Cf. the properties (SP2), (SP3) (and (SP0)) respectively, in [20].)

Proof 1. Denote the inclusions $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ by i and i', respectively. Let $\tilde{i}$ and $\tilde{i}'$ be the corresponding lifts defined in 2.28.4 and let $\tilde{f} : \tilde{X}_Z \to \tilde{X}'_{Z'}$ and $\tilde{f}_Z : Z_R \to Z'_R$ be the lifts of f and f_Z, respectively, as in 2.30.1. By 2.31.2, $\tilde{f}$ and $\tilde{f}_Z$ are also proper.

Now consider the following diagram:

$$\begin{array}{ccccc}
\tilde{i}'^*_s\,\mathrm{sp}_{\tilde{X}'_{Z'}} f_! & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{Z'_R}\, i'^* f_! & \xrightarrow{(2.10)^{-1}} & i'^* f_! \\
\mathrm{bc}_! \uparrow \cong & & (1.12) \downarrow \cong & & \cong \downarrow (1.12) \\
\tilde{i}'^*_s \tilde{f}_{s!}\,\mathrm{sp}_{\tilde{X}_Z} & & \mathrm{sp}_{Z'_R} f_{Z!}\, i^* & \xrightarrow{(2.10)^{-1}} & f_{Z!}\, i^* \\
(1.12) \downarrow \cong & & \mathrm{bc}_! \uparrow \cong & & \Vert \\
f_{Z!}\,\tilde{i}^*_s\,\mathrm{sp}_{\tilde{X}_Z} & \xrightarrow[\mathrm{bc}^*]{} & f_{Z!}\,\mathrm{sp}_{Z_R}\, i^* & \xrightarrow{(2.10)^{-1}} & f_{Z!}\, i^*
\end{array}$$

(Here we have identified $(\tilde{f}_Z)_s$ with f_Z.) By 2.19.3, the left half of this diagram commutes, as does the right lower square by 2.22, while the right upper square is clearly commutative. Applying the whole diagram to $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and taking the cone of the top and bottom row yields the claim.

2. The smooth case is proved in exactly the same way. □

The proof of 2.35 is by reducing to the special case $X = \operatorname{Spec}(k[T]) = \mathbb{A}^1$, $Z = \operatorname{Spec}(k[T]/(T^r))$, some $r \in \mathbb{N}$, and $\mathscr{F} = \Lambda_X$. We will deduce this from the fact that X is "contractible to Z".

Definition 2.39 Let X be a scheme and Z a closed subscheme of X. X is said to be *contractible to Z* if there exists a morphism $H : X \times \mathbb{A}^1 \to X$ (called a *contraction*) such that

1. $H|_{X\times\{1\}} = \mathbb{1}_X$;
2. the image of $X \times \{0\}$ and $Z \times \mathbb{A}^1$ under H is scheme-theoretically contained in Z;
3. $H|_{Z_{\mathrm{red}}\times\mathbb{A}^1}$ is the projection $Z_{\mathrm{red}} \times \mathbb{A}^1 \to Z_{\mathrm{red}}$.

Example 2.40 Let $X = \operatorname{Spec}(k[T])$ and $Z = \operatorname{Spec}(k[T]/(T^r))$ as above. Clearly, the map $H : X \times \mathbb{A}^1 \to X$ defined by $T \mapsto T{\cdot}S$, where the second factor in the source, $\mathbb{A}^1$, is identified with $\operatorname{Spec}(k[S])$, is a contraction of X to Z.

Lemma 2.41 $C_{X,Z}(\mathscr{F}) = 0$ *if X is contractible to Z and* $\mathscr{F} = \Lambda_X$.

Proof Let $H : X \times \mathbb{A}^1 \to X$ be a contraction of X to Z. For each $a \in k$, denote the inclusion $X \cong X \times \{a\} \hookrightarrow X \times \mathbb{A}^1$ by i_a, and set $H_a = H \circ i_a : X \to X$. Then $H_1 = \mathbb{1}_X$ and H_0 factors through $j : Z \hookrightarrow X$.

Fix some $a \in k$. Notice that the morphism $H_a : (X, Z) \to (X, Z)$ may be considered as a morphism of the category **ahc** introduced in 2.29 hence, by 2.30.1, there is a lift $\tilde{H}_a : \tilde{X}_Z \to \tilde{X}_Z$. Since over the s-fiber $\tilde{H}_{as}|_{Z_{\mathrm{red}}} = H_a|_{Z_{\mathrm{red}}} = \mathbb{1}_{Z_{\mathrm{red}}}$ (cf. 2.30.3), $(\tilde{H}_{as}|_Z)^*$ is also the identity. The base-change morphism bc^* corresponding to H_a thus induces a map

$$\begin{aligned}\varphi_a : \mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z = (\tilde{H}_{as}|_Z)^*\left(\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z\right) &\xrightarrow{\cong} \left(\tilde{H}_{as}^*\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)\right)|_Z \\ &\xrightarrow{\mathrm{bc}^*} \mathrm{sp}_{\tilde{X}_Z}(H_a^*\Lambda_X)|_Z \xrightarrow{\cong} \mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z.\end{aligned}$$

φ_1 is the identity morphism and φ_0 factors through (2.20): $\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z \to \Lambda_Z$. Indeed, the decomposition of H_0 in **ahc**,

$$H_0 : (X, Z) \xrightarrow{h_0} (Z, Z) \xrightarrow{j} (X, Z),$$

induces a decomposition $\tilde{H}_0 : \tilde{X}_Z \overset{\tilde{h}_0}{\to} Z_R \overset{\tilde{j}}{\to} \tilde{X}_Z$. Since the base change morphisms are compatible with composition (cf. 2.19.2), the following diagram commutes:

$$\begin{array}{ccc}
\left(\tilde{H}_{0s}^*\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)\right)|_Z & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z \\
\| & & \uparrow{\scriptstyle \mathrm{bc}^*} \\
\tilde{h}_{0s}^*\left(\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z\right)|_Z & \xrightarrow{\mathrm{bc}^*} & \tilde{h}_{0s}^*\mathrm{sp}_{Z_R}(\Lambda_Z)|_Z \\
\| & & \| \\
\mathrm{sp}_{\tilde{X}_Z}(\Lambda_X)|_Z & \xrightarrow{\mathrm{bc}^*} & \mathrm{sp}_{Z_R}(\Lambda_Z)
\end{array}$$

Now the top horizontal arrow is φ_0 and the bottom horizontal arrow is (2.10) ∘ (2.20) hence we have the claimed factorization of φ_0.

Suppose we can prove that φ_0 is the identity morphism. Then we see from its factorization just proved that (2.20) is injective. But Λ being finite, surjectivity is then immediate. Hence it suffices to prove that φ_a is independent of a. To this end, consider the morphism

$$\varphi : \pi_Z^*(\mathrm{sp}_{\tilde X_Z}(\Lambda_X)|_Z) = \big(\tilde H_s^* \mathrm{sp}_{\tilde X_Z}(\Lambda_X)\big)|_{Z\times\mathbb{A}^1} \xrightarrow{\mathrm{bc}^*}$$
$$\mathrm{sp}_{\widetilde{X\times\mathbb{A}^1}_{Z\times\mathbb{A}^1}}(\Lambda_{X\times\mathbb{A}^1})|_{Z\times\mathbb{A}^1} \xrightarrow[\cong]{(\mathrm{bc}^*)^{-1}} \big(\tilde\pi_s^*\mathrm{sp}_{\tilde X_Z}(\Lambda_X)\big)|_{Z\times\mathbb{A}^1} \cong \pi_Z^*(\mathrm{sp}_{\tilde X_Z}(\Lambda_X)|_Z),$$

where $\pi : (X\times\mathbb{A}^1, Z\times\mathbb{A}^1) \to (X,Z)$ is the canonical projection and $\pi_Z : Z\times\mathbb{A}^1 \to Z$ its restriction, and where the base change morphism corresponding to π is an isomorphism because π is smooth (2.19.1). As the notation suggests, the fiber of φ over $a\in k$ is φ_a. Indeed, the top row in the following diagram is the fiber of φ over a, the bottom row is φ_a:

$$\begin{array}{ccccccccc}
j_a^*\pi_Z^*\tilde j_s^*\mathrm{sp}(\Lambda) & \xrightarrow[\cong]{} & j_a^*\tilde i_s^*\tilde H_s^*\mathrm{sp}(\Lambda) & \xrightarrow{\mathrm{bc}^*} & j_a^*\tilde i_s^*\mathrm{sp}(\Lambda) & \xrightarrow[\cong]{(\mathrm{bc}^*)^{-1}} & j_a^*\tilde i_s^*\tilde\pi_s^*\mathrm{sp}(\Lambda) & \xrightarrow[\cong]{} & j_a^*\pi_Z^*\tilde j_s^*\mathrm{sp}(\Lambda) \\
\Big\downarrow\cong & & \Big\downarrow\cong & & \Big\downarrow\cong & & \Big\downarrow\cong & & \Big\downarrow\cong \\
 & & \tilde j_s^*\tilde i_{as}^*\tilde H_s^*\mathrm{sp}(\Lambda) & \xrightarrow{\mathrm{bc}^*} & \tilde j_s^*\tilde i_{as}^*\mathrm{sp}(\Lambda) & \xrightarrow[\cong]{(\mathrm{bc}^*)^{-1}} & \tilde j_s^*\tilde i_{as}^*\tilde\pi_s^*\mathrm{sp}(\Lambda) & & \\
 & & \Big\downarrow\cong & & & \searrow^{\mathrm{bc}^*} & \Big\downarrow\cong & & \\
\tilde j_s^*\mathrm{sp}(\Lambda) & \xrightarrow[\cong]{} & \tilde j_s^*\tilde H_{as}^*\mathrm{sp}(\Lambda) & & \xrightarrow{\mathrm{bc}^*} & & \tilde j_s^*\mathrm{sp}(\Lambda) & = & \tilde j_s^*\mathrm{sp}(\Lambda)
\end{array}$$

(Here we have written i and j_a for the inclusions $Z\times\mathbb{A}^1 \hookrightarrow X\times\mathbb{A}^1$ and $Z\times\{a\}\hookrightarrow Z\times\mathbb{A}^1$, respectively.) The trapezoid commutes by the compatibility of the base change morphism bc^* with composition (2.19.2) and the triangle commutes because the base change morphism corresponding to $\mathbb{1}_{X\times\mathbb{A}^1}$ decomposes as

$$\mathrm{sp}(\Lambda) = \tilde i_{as}^*\tilde\pi_s^*\mathrm{sp}(\Lambda) \xrightarrow{\mathrm{bc}^*} \tilde i_{as}^*\mathrm{sp}(\Lambda) \xrightarrow{\mathrm{bc}^*} \mathrm{sp}(\Lambda),$$

again by the compatibility of the base change morphism with composition. All the inner squares clearly commute hence so does the whole diagram.

Now the claim follows from the next lemma. □

Lemma 2.42 *Let X and Y be two schemes and assume that X is connected. Fix $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(Y)$ and $\varphi \in \mathrm{Hom}(\Lambda_X\boxtimes\mathscr{A}, \Lambda_X\boxtimes\mathscr{B})$. For every section $j : k\to X$ there is an induced morphism $\varphi_j \in \mathrm{Hom}(\mathscr{A},\mathscr{B})$ over $k\times Y\cong Y$. The claim is that φ_j is independent of j.*

Proof Let's fix our notation as follows: $p : X\times Y\to X$, $q : X\times Y\to Y$, $i : Y\cong k\times Y\hookrightarrow X\times Y$. Moreover, we denote (in this proof only) the bifunctor $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$ by $[-,-]$.

φ_j is the image of φ under the composition

$$\mathrm{Hom}(\Lambda\boxtimes\mathscr{A},\Lambda\boxtimes\mathscr{B}) \xrightarrow{i^*} \mathrm{Hom}(i^*(\Lambda\boxtimes\mathscr{A}), i^*(\Lambda\boxtimes\mathscr{B})) \xrightarrow{\cong} \mathrm{Hom}(\mathscr{A},\mathscr{B}),$$

which is clearly induced by the composition

$$\pi_{X\times Y*}[\Lambda\boxtimes\mathscr{A},\Lambda\boxtimes\mathscr{B}] \xrightarrow{\mathrm{adj}} \pi_{X\times Y*}i_*i^*[\Lambda\boxtimes\mathscr{A},\Lambda\boxtimes\mathscr{B}] \longrightarrow$$
$$\pi_{Y*}[i^*(\Lambda\boxtimes\mathscr{A}), i^*(\Lambda\boxtimes\mathscr{B})] \xrightarrow{\cong} \pi_{Y*}[\mathscr{A},\mathscr{B}] \quad (2.21)$$

by applying $\mathrm{H}^0 \circ \mathbf{R}\Gamma$. Now consider the following diagram where (2.21) appears as the leftmost column:

$$
\begin{array}{ccccc}
\pi_{X\times Y*}[\Lambda \boxtimes \mathscr{A}, \Lambda \boxtimes \mathscr{B}] & \xleftarrow{\cong} & \pi_{X\times Y*}(\Lambda \boxtimes [\mathscr{A}, \mathscr{B}]) & \xleftarrow{\cong} & \pi_{X*}\Lambda \otimes \pi_{Y*}[\mathscr{A}, \mathscr{B}] \\
\downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} & (2) & \downarrow{\scriptstyle \mathrm{adj}\otimes \mathbb{1}} \\
\pi_{X\times Y*} i_* i^*[\Lambda \boxtimes \mathscr{A}, \Lambda \boxtimes \mathscr{B}] & \xleftarrow[\cong]{} & \pi_{X\times Y*} i_* i^*(\Lambda \boxtimes [\mathscr{A}, \mathscr{B}]) & \xleftarrow[\cong]{} & \pi_{X*} j_* j^* \Lambda \otimes \pi_{Y*}[\mathscr{A}, \mathscr{B}] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\
\pi_{Y*}[i^*(\Lambda \boxtimes \mathscr{A}), i^*(\Lambda \boxtimes \mathscr{B})] & & \pi_{Y*}(\Lambda \otimes [\mathscr{A}, \mathscr{B}]) & (3) & \Lambda \otimes \pi_{Y*}[\mathscr{A}, \mathscr{B}] \\
\downarrow{\scriptstyle \cong} & (1) & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\pi_{Y*}[\mathscr{A}, \mathscr{B}] & = & \pi_{Y*}[\mathscr{A}, \mathscr{B}] & = & \pi_{Y*}[\mathscr{A}, \mathscr{B}]
\end{array}
$$

Here the two upper horizontal arrows on the left are induced by (1.21) and the canonical isomorphism $\mathbf{R}\underline{\mathrm{Hom}}(\Lambda, \Lambda) \cong \Lambda$. Clearly, the left upper square is commutative. The two upper horizontal arrows on the right are induced by the isomorphism in [6, Th. finitude, 1.11] which we will make explicit below. All the unlabeled vertical arrows are the canonical ones.

We will show below the commutativity of (1), (2) and (3). From this it will follow that the map $\varphi \mapsto \varphi_j$ is induced by the composition

$$\mathbf{R}\Gamma(X, \Lambda) \xrightarrow{\mathrm{adj}} \mathbf{R}\Gamma(k, \Lambda) = \Lambda$$

which is clearly independent of j since X is connected. Hence the commutativity of the diagram implies the claim of the lemma.

(1) We decompose (1) as follows (where we abstain from writing π_{Y*}):

$$
\begin{array}{ccccc}
i^*[\Lambda \boxtimes \mathscr{A}, \Lambda \boxtimes \mathscr{B}] & \xleftarrow{(1.20)} & i^*([p^*\Lambda, p^*\Lambda] \otimes [q^*\mathscr{A}, q^*\mathscr{B}]) & \longleftarrow & i^*(\Lambda \boxtimes [\mathscr{A}, \mathscr{B}]) \\
\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle t_{i*}} & (d) & {\scriptstyle t_{i*}}\downarrow{\scriptstyle \cong} \\
[i^*(\Lambda \boxtimes \mathscr{A}), i^*(\Lambda \boxtimes \mathscr{B})] & (a) & i^*[p^*\Lambda, p^*\Lambda] \otimes i^*[q^*\mathscr{A}, q^*\mathscr{B}] & \longleftarrow & i^*p^*\Lambda \otimes i^*q^*[\mathscr{A}, \mathscr{B}] \\
{\scriptstyle \cong}\downarrow{\scriptstyle t_{i*}} & & \downarrow & (e) & \downarrow{\scriptstyle \cong} \\
[i^*p^*\Lambda \otimes i^*q^*\mathscr{A}, i^*p^*\Lambda \otimes i^*q^*\mathscr{B}] & \xleftarrow{(1.20)} & [i^*p^*\Lambda, i^*p^*\Lambda] \otimes [i^*q^*\mathscr{A}, i^*q^*\mathscr{B}] & & i^*p^*\Lambda \otimes [\mathscr{A}, \mathscr{B}] \\
{\scriptstyle \cong}\downarrow & (b) & {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
[\Lambda \otimes \mathscr{A}, \Lambda \otimes \mathscr{B}] & \xleftarrow[(1.20)]{} & [\Lambda, \Lambda] \otimes [\mathscr{A}, \mathscr{B}] & \xrightarrow[\mathrm{ev}\otimes \mathbb{1}]{\cong} & \Lambda \otimes [\mathscr{A}, \mathscr{B}] \\
{\scriptstyle \cong}\downarrow & & (c) & & \downarrow{\scriptstyle \cong} \\
[\mathscr{A}, \mathscr{B}] & & = & & [\mathscr{A}, \mathscr{B}]
\end{array}
$$

Here the two top horizontal arrows on the right are induced by the canonical isomorphism $\Lambda \cong \mathbf{R}\underline{\mathrm{Hom}}(\Lambda, \Lambda)$ followed by (1.19) $\otimes$ (1.19) as in the definition of (1.21). Clearly then, (d) is commutative. The commutativity of (a) is proved in [14, p. 85], the commutativity of (b) follows from the naturality of (1.20) while (c) is obviously commutative. Finally, the commutativity of (e) may be checked on each tensor factor separately, which is easy.

② Let us abbreviate $\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A},\mathscr{B})$ by $\mathscr{R}$. Then ② may be expanded as follows:

$$\begin{CD}
\pi_{Y*}q_*(p^*\Lambda\otimes q^*\mathscr{R}) @<{\text{proj}}<{\cong}< \pi_{Y*}(q_*p^*\Lambda\otimes\mathscr{R}) @<{\text{bc (1.12)}}<< \pi_{Y*}(\pi_Y^*\pi_{X*}\Lambda\otimes\mathscr{R}) @<{\text{proj}}<{\cong}< \pi_{X*}\Lambda\otimes\pi_{Y*}\mathscr{R}\\
@V{\text{adj}}VV @V{\text{adj}}VV @V{\text{adj}}VV @V{\text{adj}}VV\\
\pi_{Y*}q_*i_*i^*(p^*\Lambda\otimes q^*\mathscr{R}) @<<{\alpha}< \pi_{Y*}(q_*i_*i^*p^*\Lambda\otimes\mathscr{R}) @<<{\beta}< \pi_{Y*}(\pi_Y^*\pi_{X*}j_*j^*\Lambda\otimes\mathscr{R}) @<{\cong}<{\text{proj}}< \pi_{X*}j_*j^*\Lambda\otimes\pi_{Y*}\mathscr{R}
\end{CD}$$

(We have implicitly made use of the isomorphism $\pi_{X\times Y*}\cong\pi_{Y*}q_*$.) The right inner square commutes by the naturality of proj (i. e. (1.3)). To define α consider the following diagram ($\mathscr{F}:=p^*\Lambda$):

$$\begin{array}{ccccccc}
q_*(\mathscr{F}\otimes q^*\mathscr{R}) & & \xleftarrow[\cong]{\text{proj}} & & & & q_*\mathscr{F}\otimes\mathscr{R}\\
\downarrow{\scriptstyle\text{adj}} & & \searrow{\scriptstyle\text{adj}} & & & & \downarrow{\scriptstyle\text{adj}}\\
q_*i_*i^*(\mathscr{F}\otimes q^*\mathscr{R}) & \xleftarrow[t_{i*}]{\cong} & q_*i_*(i^*\mathscr{F}\otimes i^*q^*\mathscr{R}) & \xleftarrow[\text{proj}]{\cong} & q_*(i_*i^*\mathscr{F}\otimes q^*\mathscr{R}) & \xleftarrow[\text{proj}]{\cong} & q_*i_*i^*\mathscr{F}\otimes\mathscr{R}
\end{array}$$

The trapezoid on the right commutes by the naturality of proj while the left triangle commutes by the definition of proj. Therefore we may define α to be the composition of the horizontal arrows in the bottom row. Since proj is compatible with composition, a simpler description of α is possible, namely as the morphism induced by the following composition:

$$q_*i_*i^*\mathscr{F}\otimes\mathscr{R}\xrightarrow{\cong} i^*\mathscr{F}\otimes i^*q^*\mathscr{R}\xrightarrow[\cong]{t_{i*}} i^*(\mathscr{F}\otimes q^*\mathscr{R})\xrightarrow{\cong} q_*i_*i^*(\mathscr{F}\otimes q^*\mathscr{R}) \tag{2.22}$$

(using the fact that $qi=\mathbb{1}_Y$).

To define β we proceed similarly. Consider the following diagram:

$$\begin{array}{ccccccc}
q_*p^* & & \xleftarrow{\text{bc (1.12)}} & & & & \pi_Y^*\pi_{X*}\\
\downarrow{\scriptstyle\text{adj}} & & \searrow{\scriptstyle\text{adj}} & & & & \downarrow{\scriptstyle\text{adj}}\\
q_*i_*i^*p^* & \xleftarrow[\cong]{} & q_*i_*\pi_Y^*j^* & \xleftarrow[\text{bc (1.12)}]{} & q_*p^*j_*j^* & \xleftarrow[\text{bc (1.12)}]{} & \pi_Y^*\pi_{X*}j_*j^*
\end{array}$$

Again, the trapezoid on the right commutes by the naturality of (1.12) while the triangle on the left commutes by the definition of (1.12). And again, since (1.12) is compatible with composition, β (defined as the bottom row) can also be described as the morphism induced by the following composition:

$$\pi_Y^*\pi_{X*}j_*j^*\xrightarrow{\cong}\pi_Y^*j^*\xrightarrow{\cong} i^*p^*\xrightarrow{\cong} q_*i_*i^*p^*. \tag{2.23}$$

③ Finally, we expand ③ as follows:

$$\begin{CD}
\pi_{Y*}q_*i_*i^*(p^*\Lambda\otimes q^*\mathscr{R}) @<{\alpha}<< \pi_{Y*}(q_*i_*i^*p^*\Lambda\otimes\mathscr{R}) @<{\beta}<< \pi_{Y*}(\pi_Y^*\pi_{X*}j_*j^*\Lambda\otimes\mathscr{R}) @<{\text{proj}}<{\cong}< \pi_{X*}j_*j^*\Lambda\otimes\pi_{Y*}\mathscr{R}\\
@V{\cong}V{t_{i*}}V @V{\cong}VV @V{\cong}VV @V{\cong}VV\\
\pi_{Y*}(i^*p^*\Lambda\otimes i^*q^*\mathscr{R}) @<{\cong}<< \pi_{Y*}(i^*p^*\Lambda\otimes\mathscr{R}) @<{\cong}<< \pi_{Y*}(\pi_Y^*j^*\Lambda\otimes\mathscr{R}) @<{\text{proj}}<{\cong}< j^*\Lambda\otimes\pi_{Y*}\mathscr{R}\\
@V{\cong}VV @. @V{\cong}VV @V{\cong}VV\\
\pi_{Y*}(\Lambda\otimes\mathscr{R}) @. @. \pi_{Y*}(\pi_Y^*\Lambda\otimes\mathscr{R}) @<{\text{proj}}<{\cong}< \Lambda\otimes\pi_{Y*}\mathscr{R}\\
@V{\cong}VV @. @V{\cong}VV @V{\cong}VV\\
\pi_{Y*}\mathscr{R} @= @= \pi_{Y*}\mathscr{R} @= \pi_{Y*}\mathscr{R}
\end{CD}$$

It will be noted that the leftmost column coincides with the rightmost column of ② so that ② and ③ really “fit together”. By the alternative descriptions of α and β given in (2.22) and (2.23), respectively, it is clear that the two top squares on the left and in the middle commute, while the commutativity of the big rectangle below is obvious. The top and middle square on the right commute by the naturality of proj thus it remains to prove the commutativity of the bottom right square. This is decomposed as follows:

$$\begin{array}{ccccccc}
\pi_{Y*}(\pi_Y^*\Lambda\otimes\mathscr{R}) & \xleftarrow[t_{\pi_Y^*}]{\text{adj}} & \pi_{Y*}(\pi_Y^*\Lambda\otimes\pi_Y^*\pi_{Y*}\mathscr{R}) & \xleftarrow{\cong} & \pi_{Y*}\pi_Y^*(\Lambda\otimes\pi_{Y*}\mathscr{R}) & \xleftarrow{\text{adj}} & \Lambda\otimes\pi_{Y*}\mathscr{R} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\pi_{Y*}\mathscr{R} & \xleftarrow[\text{adj}]{} & \pi_{Y*}\pi_Y^*\pi_{Y*}\mathscr{R} & = & \pi_{Y*}\pi_Y^*\pi_{Y*}\mathscr{R} & \xleftarrow[\text{adj}]{} & \pi_{Y*}\mathscr{R}
\end{array}$$

The two outer squares commute by the naturality of the vertical isomorphisms, while the commutativity of the middle square has been remarked in 1.(**i**). Of course, the composition of the lower horizontal arrows is the identity. □

We now start with the reduction steps in the proof of 2.35.

Step 1 *We may assume* $\mathscr{F}|_Z = 0$.

Proof Notice first that in the case $X = Z$ the base change morphism bc^* is just the identity morphism from sp_{X_R} to itself. By 2.21, the morphism (2.20) is thus an isomorphism in this case. Applying 2.38.1 to the closed embedding i yields

$$C_{X,Z}(i_!i^*\mathscr{F}) \cong \mathbb{1}_{Z!}C_{Z,Z}(i^*\mathscr{F}) = 0.$$

In view of the distinguished triangle $j_!j^*\mathscr{F} \to \mathscr{F} \to i_!i^*\mathscr{F} \to^+$ the claim follows from 2.37.1. □

Step 2 *We may assume that Z is defined by an invertible sheaf and* $\mathscr{F}|_Z = 0$.

Proof Let $f : X' \to X$ be the blow-up of X along Z. Since $f_U := f|_{X'\setminus f^{-1}(Z)} : X'\setminus f^{-1}(Z) \to U$ is an isomorphism and since $\mathscr{F}|_Z = 0$ (by step 1), we have (denoting by i' (resp. j') the inclusion of $f^{-1}(Z)$ (resp. $X'\setminus f^{-1}(Z)$) into X', and by f_Z the restriction $f|_{f^{-1}(Z)}$):

$$i^*f_!f^*\mathscr{F} \cong f_{Z!}i'^*f^*\mathscr{F} \cong f_{Z!}f_Z^*i^*\mathscr{F} = 0,$$

where the second isomorphism comes from (1.13). We deduce

$$f_!f^*\mathscr{F} \cong j_!j^*f_!f^*\mathscr{F} \cong j_!f_{U!}j'^*f^*\mathscr{F} \cong j_!f_{U!}f_U^*j^*\mathscr{F} \cong j_!j^*\mathscr{F} \cong \mathscr{F},$$

the second isomorphism again coming from (1.13). Applying 2.38.1 to f provides us with the isomorphism

$$C_{X,Z}(\mathscr{F}) \cong C_{X,Z}(f_!f^*\mathscr{F}) \cong f_{Z!}C_{X',f^{-1}(Z)}(f^*\mathscr{F}).$$

Therefore it suffices to prove $C_{X',f^{-1}(Z)}(f^*\mathscr{F}) = 0$. □

Step 3 *We may assume* $X = \mathbb{A}^n$, $Z = \mathbb{A}^{n-1} \times \{0\}$, *some* $n \geq 1$, *and* $\mathscr{F}|_Z = 0$.

Proof Since the assertion of 2.35 is local on X and Z (use 2.38.2) we may assume X to be affine and Z defined by a principal ideal (by step 2), say $X = \mathrm{Spec}(A)$, $Z = (z)$, $z \in A$. Choose an epimorphism

$k[T_1, \ldots, T_{n-1}] \twoheadrightarrow A$ and extend it to an epimorphism $k[T_1, \ldots, T_n] \twoheadrightarrow A$ by $T_n \mapsto z$. This defines a closed embedding $f : X \hookrightarrow \mathbb{A}^n$ such that

$$f^{-1}(\mathbb{A}^{n-1} \times \{0\}) = f^{-1}(\mathrm{Var}(T_n)) = \mathrm{Var}(T_n \cdot A) = \mathrm{Var}((z)) = Z.$$

Applying 2.38.1 to this closed embedding gives

$$C_{\mathbb{A}^n, \mathbb{A}^{n-1} \times \{0\}}(f_! \mathscr{F}) \cong f_{Z!} C_{X,Z}(\mathscr{F}),$$

where $f_Z = f|_Z : Z \hookrightarrow \mathbb{A}^{n-1} \times \{0\}$. If the left hand side is 0 then so is $C_{X,Z}(\mathscr{F})$ since f_Z is a closed immersion. □

Step 4 *2.35 is implied by the following statement: $C_{\mathbb{A}^n, \mathbb{A}^{n-1} \times \{0\}}(\mathscr{F})$ vanishes at the generic point of $\mathbb{A}^{n-1} \times \{0\}$ for all $n \geq 1$ and for all $\mathscr{F}$ such that $\mathscr{F}|_Z = 0$.*

Proof By the previous step, we have to prove 2.35 for $X = \mathbb{A}^n$, $Z = \mathbb{A}^{n-1} \times \{0\}$ and $\mathscr{F}|_Z = 0$. We do this by induction on n. In case $n = 1$, Z is a point, thus the claim follows from the assumption that $C_{X,Z}(\mathscr{F})$ vanishes generically.

For the induction step let $Y \subset Z$ be the closure of the support of $C_{X,Z}(\mathscr{F})$. By the assumption that $C_{X,Z}(\mathscr{F})$ vanishes generically, we have $Y \neq Z$, hence the Noether normalization lemma ensures the existence of a line $\tau \subset Z \subset X$ such that the projection $q : X \to X/\tau$ restricts to a finite morphism $q|_Y$. Set $X' = X/\tau \cong \mathbb{A}^{n-1}$, $Z' = Z/\tau \cong \mathbb{A}^{n-2} \times \{0\}$, and think of X as a line bundle over X'. Then q extends to the projective closure $\overline{X}$ of X (defined as in [11, 8.4]); i. e. we get a compactification

$$X \xrightarrow{e} \overline{X} \xrightarrow{\overline{q}} X'$$

of q. Set $\overline{Z} = \overline{q}^{-1}(Z')$. Furthermore, let $\overline{\mathscr{F}} \in \mathfrak{D}^b_{ctf}(\overline{X})$ be the extension of $\mathscr{F}$ by zero, i. e. $\overline{\mathscr{F}} = e_! \mathscr{F}$. Then the triple $(X', Z', \overline{q}_! \overline{\mathscr{F}})$ satisfies the induction hypothesis, thus together with 2.38.1 we get

$$(\overline{q}|_{\overline{Z}})_! C_{\overline{X}, \overline{Z}}(\overline{\mathscr{F}}) \cong C_{X', Z'}(\overline{q}_! \overline{\mathscr{F}}) = 0.$$

Now, $C_{\overline{X}, \overline{Z}}(\overline{\mathscr{F}})$ has support on $\overline{Y} := Y \cup (\overline{Z} \backslash Z)$. Indeed, for any $x \in Z \backslash Y \subset \overline{Z}$ we have

$$C_{\overline{X}, \overline{Z}}(\overline{\mathscr{F}})_{\overline{x}} \cong (e_Z^* C_{\overline{X}, \overline{Z}}(\overline{\mathscr{F}}))_{\overline{x}} \cong C_{X,Z}(e^* e_! \mathscr{F})_{\overline{x}} \cong C_{X,Z}(\mathscr{F})_{\overline{x}} = 0, \qquad (2.24)$$

where the second isomorphism exists by 2.38.2 (as before, $e_Z = e|_Z : Z \hookrightarrow \overline{Z}$). Moreover, $\overline{q}|_{\overline{Y}}$ is still finite. Indeed, it is quasi-finite over Y by construction, and proper. But $\overline{Z} \backslash Z$ lies in $\overline{X} \backslash X$ which is mapped isomorphically to X' hence $\overline{q}$ is quasi-finite on $\overline{Z} \backslash Z$ as well. This clearly implies $C_{\overline{X}, \overline{Z}}(\overline{\mathscr{F}}) = 0$ hence also $C_{X,Z}(\mathscr{F}) = 0$ by the same isomorphisms as in (2.24). □

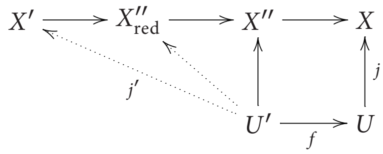
Step 5 *We may further reduce to the following statement: Assume that X is normal, Z is defined by a locally principal sheaf of ideals and $\mathscr{F} = j_! \mathscr{H}$, some constant (finite) sheaf $\mathscr{H}$ on U. Then $C_{X,Z}(\mathscr{F})$ vanishes at each generic point of Z.*

Proof In fact, we will prove that this statement implies the same statement with no conditions on $\mathscr{F}$ (except $\mathscr{F}|_Z = 0$). Clearly this suffices by step 4.

Thus let X be normal, Z a locally principal closed subscheme, and $\mathscr{F}|_Z = 0$. We may clearly assume X to be connected (2.38.2) and thus irreducible (since X is normal). Also we may assume Z to be irreducible. Using 2.37.1, a standard homological algebra argument reduces us to the case where $\mathscr{F}$ is a complex concentrated in degree $d \in \mathbb{Z}$, i. e. where $\mathscr{F}$ is a sheaf. Since $\mathscr{F}$ is constructible and

X irreducible, there exists a non-empty open subset $V' \subset U$ such that $\mathscr{F}|_{V'}$ is locally constant. The lemma below (2.43.2) implies that replacing X by an open neighborhood in X of the generic point of Z, we may assume $\mathscr{F} = j_!\mathscr{G}$, some locally constant constructible sheaf $\mathscr{G}$ on U. (Apply the lemma to an open affine neighborhood $\mathrm{Spec}(A)$ in X of the generic point of Z such that $Z \cap \mathrm{Spec}(A)$ corresponds to (z), and $V = V' \cap \mathrm{Spec}(A)$. The sought-after open subset is then $\mathrm{Spec}(A_x)$.)

We now claim that the cohomology sheaf $\mathrm{H}^m(C_{X,Z}(\mathscr{F}))$ vanishes generically for all $m \in \mathbb{Z}$ and all $\mathscr{F}$ of the form above. This is obviously true if $m < d$. Suppose it is true for all $r < m$, some $m \in \mathbb{Z}$. Let $f : U' \to U$ be a finite étale map such that $f^*\mathscr{G}$ is constant (we may assume U' connected), and let $U' \hookrightarrow X'' \to X$ be a compactification of jf with $X'' \to X$ finite. Notice that U' is normal hence irreducible hence so is X''. Moreover U' factors uniquely through $X''_{\mathrm{red}} \hookrightarrow X''$ and then also through the normalization $X' \to X''_{\mathrm{red}}$ as in the following commutative diagram:



Denote the composition of the arrows in the top row by $f' : X' \to X$. This is a finite map. Also, j' is an open immersion which implies that the square $f'j' = jf$ is cartesian. Hence $f'^*\mathscr{F} = f'^*j_!\mathscr{G} \cong j'_!f^*\mathscr{G}$ by the base change isomorphism (1.13). Notice also that $Z' := f'^{-1}(Z) \subset X'$ is again defined by a locally principal sheaf of ideals hence by our assumption $C_{X',Z'}(f'^*\mathscr{F})$ vanishes generically. But the generic point of Z is hit only by generic points of Z' and consequently Lemma 2.38.1 implies that also $C_{X,Z}(f'_!f'^*\mathscr{F})$ vanishes generically.

Set $\mathscr{G}'$ to be the cokernel of the injective morphism $\mathrm{adj} : \mathscr{G} \hookrightarrow f_!f^*\mathscr{G}$. $\mathscr{G}'$ is again a local system and $j_!\mathscr{G}'$ is isomorphic to the cokernel of the injective morphism $\mathscr{F} = j_!\mathscr{G} \hookrightarrow j_!f_!f^*\mathscr{G} \cong f'_!j'_!f^*\mathscr{G} \cong f'_!f'^*\mathscr{F}$. The associated distinguished triangle satisfies the hypotheses of 2.37.1 with ξ the generic point of Z by induction hypothesis. Hence $\mathrm{H}^m(C_{X,Z}(\mathscr{F}))$ vanishes generically. □

Lemma 2.43 *Let A be a normal noetherian domain and let $z \in A$ such that $\sqrt{(z)} \subset A$ is a prime ideal. Then:*

1. *There exist $w, t \in A$, and $r \in \mathbb{N}$ such that, in A_w, $(z) = (t^r)$ and $(t) \subset A_w$ is a prime ideal.*

2. *Given any non-empty open subset $V \subset \mathrm{Spec}(A_z)$ there exists $x \in A \backslash \sqrt{(z)}$ such that $\mathrm{Spec}(A_{xz}) \subset V$.*

Proof If $z = 0$ then both claims are obvious. Hence we may assume $z \in A \backslash \{0\}$. Set $\mathfrak{p} = \sqrt{(z)}$.

1. By Krull's Hauptidealsatz, $\mathfrak{p}$ is a prime ideal of height 1. Let $p_1, \ldots, p_n$ be a set of generators for $\mathfrak{p}$. The normality of A implies that $A_{\mathfrak{p}}$ is a discrete valuation ring with uniformizer, say, $t \in \mathfrak{p}$. For each $i = 1, \ldots, n$ there exist $r_i \in \mathbb{N}$, $u_i, v_i \in A \backslash \mathfrak{p}$ such that $p_i u_i = t^{r_i} v_i$. Similarly, there exist $r \in \mathbb{N}$, $u, v \in A \backslash \mathfrak{p}$ such that $zu = t^r v$. Set $w = uv \prod_{i=1}^n u_i \in A \backslash \mathfrak{p}$.

2. Choose w, t, r as in part 1 and let $g \in A$ such that $\varnothing \neq \mathrm{Spec}(A_g) \subset V$. In particular, $\mathrm{Spec}(A_{wg}) \subset \mathrm{Spec}(A_{wt})$ hence, as above, there exist $s \geq 1$, $u, v \in A \backslash \mathfrak{p}$ such that $gu = t^s v w^n$ in A_w, some $n \in \mathbb{Z}$. Set $x = uvw \in A \backslash \mathfrak{p}$. □

We are now ready to finish the proof of 2.35.

Step 6 *Assume that X is normal, Z is defined by a locally principal sheaf of ideals and $\mathscr{F} = j_!\mathscr{G}$, some constant (finite) sheaf $\mathscr{G}$ on U. Then $C_{X,Z}(\mathscr{F})$ vanishes at each generic point of Z. Hence 2.35 is true.*

Proof Doing induction as in the previous step and using a short exact sequence $K \hookrightarrow \Lambda_U^n \twoheadrightarrow \mathscr{G}$ of sheaves on U (some n, K is necessarily constant), one easily reduces to the case $\mathscr{G} = \Lambda_U$.

Also as in the previous step, we may assume X and Z to be irreducible. Let ξ be the generic point of Z (which we may assume to be different from the generic point of X—otherwise the claim being trivial) and let $V \subset X$ be the set of regular points of X. V is an open subset since X is an excellent scheme ([12, 7.8.6]), and contains ξ since $\mathscr{O}_{X,\xi}$ is normal and has dimension 1. Replacing X by V and Z by $Z \cap V$ we may thus assume X to be smooth over k.

Similarly, Z_{red} is an integral excellent scheme hence there is an open subset $V \subset X$ containing ξ such that both V and $Z_{\text{red}} \cap V$ are smooth over k. Replace X by V and Z by $Z \cap V$.

By the lemma below (2.44), there exist an open neighborhood V of ξ in X, a natural number $n \geq 1$, and an étale morphism $V \to \mathbb{A}^n$ such that the inverse image of $\mathbb{A}^{n-1} \times \{0\}$ is equal to $Z_{\text{red}} \cap V$. Composing with the (smooth) projection $\mathbb{A}^n \to \mathbb{A}^1$ onto the last component and replacing X by V, Z by $Z \cap V$, we get a smooth morphism $f : X \to \mathbb{A}^1$ such that $f^{-1}(0) = Z_{\text{red}}$.

Next, we may assume $X = \operatorname{Spec}(A)$ to be affine, A a normal integral k-algebra, such that Z corresponds to the principal ideal (z) generated by some $z \in A$. Let $a = f^\sharp(T)$, where we identify $\mathbb{A}^1$ with $\operatorname{Spec}(k[T])$. Thus we have $\sqrt{(z)} = (a)$, and a is prime. The lemma above (2.43.1) implies that, replacing A by a suitable localization A_w, $w \in A \backslash \sqrt{(z)}$, we may assume $(z) = (a^r)$, i. e. $f^{-1}(\operatorname{Var}(T^r)) = Z$. We conclude that

$$
\begin{aligned}
C_{X,Z}(\mathscr{F})_{\overline{\xi}} &= C_{X,Z}(j_!\Lambda_U)_{\overline{\xi}} && \\
&\cong C_{X,Z}(\Lambda_X)_{\overline{\xi}} && \text{since } C_{X,Z}(i_*\Lambda_Z) = 0 \\
&\cong C_{X,Z}(f^*\Lambda_{\mathbb{A}^1})_{\overline{\xi}} && \\
&\cong (f^* C_{\mathbb{A}^1,\operatorname{Var}(T^r)}(\Lambda_{\mathbb{A}^1}))_{\overline{\xi}} && \text{by 2.38.2} \\
&\cong C_{\mathbb{A}^1,\operatorname{Var}(T^r)}(\Lambda_{\mathbb{A}^1})_0 && \\
&= 0 && \text{by 2.41 and 2.40.}
\end{aligned}
$$

□

Lemma 2.44 *Let Z be a closed subscheme of a scheme X, and $x \in X$ such that both Z and X are smooth over k at x. Moreover, let r be the minimal number of generators for the ideal $\mathscr{I}_{Z,x} \subset \mathscr{O}_{X,x}$. Then there exist an open neighborhood V of x in X, a natural number $n \geq r$, and an étale morphism $V \to \mathbb{A}^n$ (over k, as usual) such that the inverse image of $\mathbb{A}^{n-r} \times \{0\}$ is equal to $Z \cap V$.*

Proof This is part of the more general statement in [13, 17.12.2]. We provide it here for convenience only. □

We can now give a proof of 2.21 as promised:

Proof (2.21) The proof is similar to the one just given. Denote by $C_X(\mathscr{F})$ the cone of (2.10) : $\mathscr{F} \to \operatorname{sp}_X(\mathscr{F})$. As in Lemma 2.38 one proves that, for a morphism $f : X \to Y$, $f_! C_X(\mathscr{F}) \cong C_Y(f_!\mathscr{F})$ if f is proper, and $f^*C_Y(\mathscr{F}) \cong C_X(f^*\mathscr{F})$ if f is smooth. It follows, as in step 3, that the question is local and we may reduce to the case $X = \mathbb{A}^n$, some n. Also, we may assume that $\mathscr{F}$ is a sheaf.

We will now prove that $C_X(\mathscr{F})$ vanishes generically. For this, we may replace X by an open subset on which $\mathscr{F}$ is locally constant. Replacing X by a suitable étale cover, we may assume that $\mathscr{F}$ is constant, and then reduce to $\mathscr{F} = \Lambda_X$ as in step 6. As π_X is smooth, it is sufficient to treat the case $X = k$, $\mathscr{F} = \Lambda_k$.

Going back to the definition of (2.10) we see that our task is to prove

$$(\Lambda_{\overline{R^h}})_s \xrightarrow{\text{adj}} (j_*j^*\Lambda_{\overline{R^h}})_s$$

an isomorphism. But since $j^*\lambda_{\overline{R^h}} \cong \Lambda_{\overline{\eta^h}}$ is a flask sheaf (recall that $\overline{\eta^h}$ is separably closed, see 1.§4), $j_*\Lambda_{\overline{\eta^h}}$ can be computed on the level of complexes where it is easily seen to coincide with $\Lambda_{\overline{R^h}}$ ($\overline{R^h}$ being henselian, see again 1.§4). Thus we have to prove an isomorphism the first map in

$$(j_*\Lambda_{\overline{\eta^h}})_s \xrightarrow{\text{adj}} (j_*j^*j_*\Lambda_{\overline{\eta^h}})_s \xrightarrow{\text{adj}} (j_*\Lambda_{\overline{\eta^h}})_s.$$

j_* (on the level of sheaves) being fully faithful, the second map is an isomorphism, while the composition of the two maps is the identity. Thus the claim. □

3 The trace map

From now on, if not otherwise mentioned, we will be working in the category $\mathbf{sCor}(k)$ of self-correspondences.

§1 Definition

To each self-correspondence $c : C \to X \times X$, to each complex of sheaves $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and to each open subset $\beta \subset \mathrm{Fix}(c)$, we will associate a trace map

$$\mathrm{tr}_{c,\mathscr{F},\beta} : \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{H}^0(\beta, K_\beta).$$

In the case $c = c_k$ and $\beta = \mathrm{Fix}(c)$ the target of this map can be identified with Λ, $\mathscr{F}$ can be identified with a bounded complex of finitely generated projective Λ-modules and the trace map will then be seen to coincide with the "usual trace map" $\mathrm{Hom}(\mathscr{F},\mathscr{F}) \to \Lambda$. For general correspondences, however, the definition of the trace map is somewhat involved.

Definition 3.1 1. We define the *global trace morphism (associated to c and $\mathscr{F}$)*,

$$\underline{\mathrm{tr}}_{c,\mathscr{F}} : \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) \longrightarrow \Delta'_* K_{\mathrm{Fix}(c)},$$

as the following composition:

$$\mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) \xrightarrow[\cong]{(1.6)} c^! \, \mathbf{R}\underline{\mathrm{Hom}}(p_1^*\mathscr{F}, p_2^!\mathscr{F}) \xrightarrow[\cong]{(1.18)} c^!(\mathbb{D}\mathscr{F} \boxtimes \mathscr{F}) \xrightarrow{\mathrm{adj}}$$
$$c^!\Delta_*\Delta^*(\mathbb{D}\mathscr{F} \boxtimes \mathscr{F}) \xrightarrow[\cong]{} c^!\Delta_*(\mathbb{D}\mathscr{F} \otimes \mathscr{F}) \xrightarrow{\mathrm{ev}} c^!\Delta_* K_X \xrightarrow[\cong]{(1.14)} \Delta'_* c'^! K_X = \Delta'_* K_{\mathrm{Fix}(c)}.$$

2. The above morphism induces the *global trace map (associated to c and $\mathscr{F}$)* on cohomology,

$$\mathrm{tr}_{c,\mathscr{F}} := \mathrm{H}^0(C, \underline{\mathrm{tr}}_{c,\mathscr{F}}) : \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}),$$

where we used the identifications

$$\mathrm{H}^0(C, \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F})) \cong \mathrm{Hom}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) \cong \mathrm{Hom}(c_{2!}c_1^*\mathscr{F}, \mathscr{F}).$$

3. If $j : \beta \hookrightarrow \mathrm{Fix}(c)$ is an open subset as before, we denote by

$$\mathrm{tr}_{c,\mathscr{F},\beta} : \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{H}^0(\beta, K_\beta)$$

the composition of $\mathrm{tr}_{c,\mathscr{F}}$ with the restriction map

$$\mathrm{res}_\beta : \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \to \mathrm{H}^0(\mathrm{Fix}(c), j_*j^*K_{\mathrm{Fix}(c)}) \cong \mathrm{H}^0(\beta, K_\beta)$$

induced by adj. This map is called the *trace map with respect to β (associated to c and $\mathscr{F}$)*.

4. Finally, if β is in addition proper over k, we may consider the composition

$$\int_\beta \mathrm{tr}_{c,\mathscr{F},\beta} : \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{tr}_{c,\mathscr{F},\beta}} \mathrm{H}^0(\beta, K_\beta) \xrightarrow{\int_\beta} \Lambda,$$

where the second map is (1.9). If β is also a connected component of $\mathrm{Fix}(c)$ then $\int_\beta \mathrm{tr}_{c,\mathscr{F},\beta}$ is denoted $\mathrm{lt}_{c,\mathscr{F},\beta}$ and called the *local term at β*.

If there is no risk of confusion we will suppress c, $\mathscr{F}$, or β in the notation introduced above, thus simply writing tr_c or lt_β etc.

Example 3.2 Let us compute the local term of a cohomological correspondence in a particularly simple situation, namely when $c = c_k : k \to k \times k$ is the trivial correspondence. In this case, $\mathrm{Fix}(c) = k$ and $\mathscr{F}$ can be identified with a bounded complex of finitely generated projective Λ-modules $(F_i)_i$ (cf. 1.(**d**)). Most of the maps in the definition of $\underline{\mathrm{tr}}$ collapse to the identity and we end up with the following trace map

$$\mathrm{tr} : \mathrm{Hom}(\mathscr{F}, \mathscr{F}) \xrightarrow{(1.18)} \mathrm{Hom}(\mathscr{F}, \Lambda) \otimes \mathscr{F} \xrightarrow{\mathrm{ev}} \Lambda,$$

which is thus equal to the one defined in [3, I, 8.1] (cf. [14, III, p. 89]). It follows ([3, I, 8.1.2]) that if the cohomological correspondence $u : \mathscr{F} \to \mathscr{F}$ has components $u_i : F_i \to F_i$ then

$$\mathrm{tr}(u) = \sum_i (-1)^i \mathrm{Tr}(u_i) \in \Lambda,$$

where Tr is the usual trace of an endomorphism of projective finite type modules.

Notation 3.3 If $u \in \mathrm{Hom}_{c_k}(\mathscr{F}, \mathscr{F})$ is a cohomological correspondence lifting the trivial correspondence c_k we will write

$$\mathrm{Tr}(u) = \sum_i (-1)^i \mathrm{Tr}(u_i) \in \Lambda$$

for the usual trace of an endomorphism of complexes as in the example. Thus we have proved above: $\mathrm{tr}(u) = \mathrm{Tr}(u)$.

Having defined the trace map, we now proceed to study its behavior with respect to the various operations on cohomological correspondences discussed in the previous section, i. e. with respect to pushforward, pullback and specialization. Suppose e. g. that $c : C \to X \times X$ is a correspondence and $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ a complex of sheaves. In 2.§4 we defined the restriction map

$$[j_W]^* : \mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) \longrightarrow \mathrm{Hom}_{c|_W}(\mathscr{F}, \mathscr{F}) \tag{3.1}$$

associated to an open subset $W \subset C$. On the other hand, the trace maps tr_c and $\mathrm{tr}_{c|_W}$ map the left and right hand side of (3.1) into $\mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)})$ and $\mathrm{H}^0(\beta, K_\beta)$ ($\beta = W \cap \mathrm{Fix}(c) = \mathrm{Fix}(c|_W)$), respectively. What is then the operation

$$\mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \longrightarrow \mathrm{H}^0(\beta, K_\beta)$$

corresponding to $[j_W]^*$? It would be nice if it was just the restriction map on cohomology, res_β, i. e. if the following diagram commuted:

$$\begin{array}{ccc} \mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) & \xrightarrow{\mathrm{tr}} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\ {\scriptstyle [j_W]^*}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_\beta} \\ \mathrm{Hom}_{c|_W}(\mathscr{F}, \mathscr{F}) & \xrightarrow[\mathrm{tr}]{} & \mathrm{H}^0(\beta, K_\beta) \end{array}$$

This turns out to be indeed true and will be proved below (3.22) Speaking informally, we might say that "the trace map is natural with respect to restriction".

Similar questions could be asked in the case of specialization or pushforward, and here also we hope that the trace map is "natural" with respect to these operations. But instead of answering these questions separately we prefer to generalize the setting until all these questions become instantiations of one general question. Having answered this general question we will then be left with interpreting the answer, a general “naturality property” of the trace map, in the three contexts mentioned above. This is, roughly, the outline of the next three paragraphs.

§2 Cohomological morphisms

Our first goal is to define a category **cor′** in which the proof of the naturality property is going to take place. It will be an artificial category, designed for this specific proof only.

Let $c : C \to X \times X$ be a self-correspondence. It gives rise to a diagram in the category of schemes as follows:

$$\begin{array}{ccccccc} \mathrm{Fix}(c) & \xrightarrow{c'} & X & & & & \\ \downarrow{\scriptstyle \Delta'} & & \downarrow{\scriptstyle \Delta} & & & & \\ C & \xrightarrow[c]{} & X \times X & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & X & \xrightarrow{\pi_X} & k \end{array}$$

This diagram generates a subcategory $\mathbf{D}(c)$ of **Sch** whose objects and morphisms will be denoted $Ob(c)$ and $Mor(c)$, respectively. Given a second correspondence $\overline{c}$ there is a canonical functor $\mathbf{D}(c) \to \mathbf{D}(\overline{c})$ which will usually be denoted $\overline{(\cdot)}$. We say that a commutative square $g_4 g_1 = g_2 g_3$ in $Mor(c)$ is *universally cartesian* if for every correspondence $\overline{c}$, $\overline{g}_4 \overline{g}_1 = \overline{g}_2 \overline{g}_3$ is cartesian as a square in $\mathbf{Sch}/k$ (e. g. the square $c\Delta' = \Delta c'$ above is universally cartesian).

Definition 3.4 Let c and $\overline{c}$ be two self-correspondences. A *cohomological premorphism from c to $\overline{c}$* is a triple

$$(\{f_Z, t_Z\}_{Z \in Ob(c)}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}_{g \in Mor(c)}, \iota),$$

where

- f_Z is a functor $\mathfrak{D}^b_{ctf}(Z) \to \mathfrak{D}^b_{ctf}(\overline{Z})$;
- t_Z is a morphism $f_Z \mathscr{A} \otimes f_Z \mathscr{B} \to f_Z(\mathscr{A} \otimes \mathscr{B})$, natural in $\mathscr{A}$ and $\mathscr{B}$;
- bc^*_g is a morphism of functors $\overline{g}^* f_{Z_2} \to f_{Z_1} g^*$, if $g : Z_1 \to Z_2$;
- $\mathrm{bc}^!_g$ is a morphism of functors $f_{Z_1} g^! \to \overline{g}^! f_{Z_2}$, if $g : Z_1 \to Z_2$;
- ι is an isomorphism $f_k \Lambda_k \to \Lambda_k$.

Two such cohomological premorphisms

$$(\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota), \qquad (\{f'_Z, t'_Z\}, \{\mathrm{bc}'^*_g, \mathrm{bc}'^!_g\}, \iota')$$

from c to $\overline{c}$ will be identified if there is, for each $Z \in Ob(c)$, a natural isomorphism of functors

$$f_Z \xrightarrow{\cong} f'_Z,$$

compatible with all the data in an obvious way. The category **cor′** has as objects self-correspondences over k, and equivalence classes of cohomological premorphisms as morphisms. Given two cohomological premorphisms

$$(\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota) : c \to \overline{c}, \qquad (\{f_{\overline{Z}}, t_{\overline{Z}}\}, \{\mathrm{bc}^*_{\overline{g}}, \mathrm{bc}^!_{\overline{g}}\}, \overline{\iota}) : \overline{c} \to \overline{\overline{c}},$$

the triple $(\{f_{\overline{Z}} \circ f_Z, t_Z \circ t_{\overline{Z}}\}, \{\mathrm{bc}^*_g \circ \mathrm{bc}^*_{\overline{g}}, \mathrm{bc}^!_{\overline{g}} \circ \mathrm{bc}^!_g\}, \overline{\iota} \circ \iota)$ clearly defines a cohomological premorphism from c to $\overline{\overline{c}}$ compatible with the above identifications, and $(\{\mathbb{1}, \mathbb{1}\}, \{\mathbb{1}, \mathbb{1}\}, \mathbb{1})$ is the identity with respect to this operation, thus defining an identity morphism.

In the sequel we will not distinguish between cohomological premorphisms and equivalence classes thereof, and speak of cohomological premorphisms in both cases. The careful reader will convince herself that all these statements remain meaningful and true when translated in order to take into account the identifications.

Notation 3.5 Let $F = (\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ be a cohomological premorphism from c to $\overline{c}$. For each $Z \in Ob(c)$ and each $g : Z_1 \to Z_2$ in $Mor(c)$, F gives rise to the following morphisms:

$$\begin{aligned} \tilde{\pi}_Z &: f_Z K_Z = f_Z \pi^!_Z \Lambda_k \xrightarrow{\mathrm{bc}^!_{\pi_Z}} \pi^!_{\overline{Z}} f_k \Lambda_k \xrightarrow[\cong]{\iota} \pi^!_{\overline{Z}} \Lambda_k = K_{\overline{Z}}, \\ r_Z &: f_Z\, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}(f_Z \mathscr{A}, f_Z \mathscr{B}) \end{aligned}$$

adjoint to

$$\begin{aligned} & f_Z\, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) \otimes f_Z \mathscr{A} \xrightarrow{t_Z} f_Z(\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) \otimes \mathscr{A}) \xrightarrow{\mathrm{ev}} f_Z \mathscr{B}, \\ d_Z &: f_Z \mathbb{D}\mathscr{A} \xrightarrow{r_Z} \mathbf{R}\underline{\mathrm{Hom}}(f_Z \mathscr{A}, f_Z K_Z) \xrightarrow{\tilde{\pi}_Z} \mathbb{D} f_Z \mathscr{A}, \\ \mathrm{bc}_{*g} &: f_{Z_2} g_* \xrightarrow{\mathrm{adj}} \overline{g}_* \overline{g}^* f_{Z_2} g_* \xrightarrow{\mathrm{bc}^*_g} \overline{g}_* f_{Z_1} g^* g_* \xrightarrow{\mathrm{adj}} \overline{g}_* f_{Z_1}, \\ \mathrm{bc}_{!g} &: \overline{g}_! f_{Z_1} \xrightarrow{\mathrm{adj}} \overline{g}_! f_{Z_1} g^! g_! \xrightarrow{\mathrm{bc}^!_g} \overline{g}_! \overline{g}^! f_{Z_2} g_! \xrightarrow{\mathrm{adj}} f_{Z_2} g_!, \\ \iota_Z &: \Lambda_{\overline{Z}} \xrightarrow{\cong} \pi^*_{\overline{Z}} \Lambda_k \xrightarrow[\cong]{\iota^{-1}} \pi^*_{\overline{Z}} f_k \Lambda_k \xrightarrow{\mathrm{bc}^*_{\pi_Z}} f_Z \pi^*_Z \Lambda_k \xrightarrow{\cong} f_Z \Lambda_Z. \end{aligned}$$

In the sequel, we will often omit the index Z of f_Z and t_Z when this causes no confusion.

Example 3.6 In this example, we will show how we intend to apply results about the category **cor′** to the situations mentioned at the end of the last paragraph, thus giving a partial justification for introducing this category.

1. Let R be a discrete valuation ring over k whose residue field is also k, and denote by **SCor** the corresponding digraph defined in 2.§5. We may define a morphism of digraphs

   $$S : \mathbf{SCor} \longrightarrow \mathbf{cor}',$$

   the *specialization functor*. S is the identity on objects, and given a lift $\tilde{c}$ of c over R, the cohomological premorphism $S(\tilde{c}) = (\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ from c to $\overline{c} := \tilde{c}_s$ is defined as follows. For each $Z \in Ob(c)$, $\tilde{c}$ defines a lift $(\tilde{Z}, \varphi)$ over R such that $\tilde{Z}_s = \overline{Z}$, and for each $g \in Mor(c)$, it defines a lift $\tilde{g}$ over R such that $\tilde{g}_s = \overline{g}$. We may thus set $f_Z = \mathrm{sp}_{\tilde{Z}}$, bc^*_g and $\mathrm{bc}^!_g$ the morphisms bc^* and $\mathrm{bc}^!$, respectively, defined in 2.§5, while $t_Z : \mathrm{sp}_{\tilde{Z}} \mathscr{A} \otimes \mathrm{sp}_{\tilde{Z}} \mathscr{B} \to \mathrm{sp}_{\tilde{Z}}(\mathscr{A} \otimes \mathscr{B})$ is the morphism

   $$\begin{aligned} i^* j_* b^* \mathscr{A} \otimes i^* j_* b^* \mathscr{B} &\xrightarrow[\cong]{t_{i^*}} i^*(j_* b^* \mathscr{A} \otimes j_* b^* \mathscr{B}) \\ &\xrightarrow{t_{j_*}} i^* j_*(b^* \mathscr{A} \otimes b^* \mathscr{B}) \\ &\xrightarrow[\cong]{t_{b^*}} i^* j_* b^*(\mathscr{A} \otimes \mathscr{B}) \end{aligned}$$

(in the notation of 1.(o) and where $b : \tilde{Z}_{\overline{\eta^h}} \to \tilde{Z}_\eta \cong Z_\eta \to Z$). Finally, for ι we take the isomorphism $(2.10)^{-1} : \mathrm{sp}_R\Lambda_k \to \Lambda_k$. Alternatively, and sometimes more usefully, it may be described as

$$i^*j_*b^*\Lambda_k \xrightarrow{\varepsilon} i^*j_*\Lambda_{\overline{\eta^h}} \xrightarrow{\varepsilon'^{-1}} i^*\Lambda_{\overline{R^h}} \xrightarrow{\varepsilon} \Lambda_k, \tag{3.2}$$

as is shown by the commutativity of the following diagram (cf. (1.7)):

$$\begin{array}{ccccccc}
i^*j_*b^*\Lambda & \xleftarrow[\cong]{} & i^*j_*j^*c^*\Lambda & \xleftarrow[\cong]{\mathrm{adj}} & i^*c^*\Lambda & \xrightarrow[\cong]{} & b^*\Lambda \\
\varepsilon\downarrow\cong & & \varepsilon\downarrow\cong & & \varepsilon\downarrow\cong & & \varepsilon\downarrow\cong \\
i^*j_*\Lambda & \xleftarrow[\varepsilon]{\cong} & i^*j_*j^*\Lambda & \xleftarrow[\mathrm{adj}]{\cong} & i^*\Lambda & \xrightarrow[\varepsilon]{\cong} & \Lambda
\end{array}$$

(Here $c : \overline{R^h} \to k$ is the structure morphism so that $cj = b$.)

2. Let **PCor** = **PCor**(k) denote the subcategory of **sCor**(k) whose morphisms are proper. We may then define a functor

$$P : \mathbf{PCor} \longrightarrow \mathbf{cor}',$$

the *proper pushforward functor*. P is the identity on objects, while given a proper morphism $[f] : c \to \overline{c}$ of correspondences, we define $P([f]) = (\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ as follows. For each $Z \in Ob(c)$, $[f]$ defines a proper morphism of schemes $[f]_Z : Z \to \overline{Z}$ such that $\overline{g}[f]_{Z_1} = [f]_{Z_2}g$ for every $g : Z_1 \to Z_2$ in $Mor(c)$. We may thus set $f_Z = [f]_{Z*} = [f]_{Z!}$, ι the identity, t_Z to be $t_{[f]_{Z*}}$ as defined in (1.2), and bc^*_g and $\mathrm{bc}^!_g$ are the usual base change morphisms (1.12) and (1.11), respectively. Functoriality of P follows from the fact that the base change morphisms and (1.2) are compatible with composition.

3. Finally, let **RCor** = **RCor**(k) denote the subcategory of **sCor**(k) whose morphisms are open immersions. We will define a functor

$$Q : \mathbf{RCor}^{\circ} \longrightarrow \mathbf{cor}',$$

the *restriction functor*. It is the identity on objects, and given an open immersion $[f] : \overline{c} \to c$ of correspondences, the cohomological premorphism $Q([f]) = (\{f_Z, t_Z\}, \{\mathrm{bc}^*, \mathrm{bc}^!\}, \iota)$ is defined as follows. For each $Z \in Ob(c)$, $[f]$ defines an open immersion of schemes $[f]_Z : \overline{Z} \to Z$ such that $[f]_{Z_2}\overline{g} = g[f]_{Z_1}$ for every $g : Z_1 \to Z_2$ in $Mor(c)$. We may thus set $f_Z = [f]^*_Z = [f]^!_Z$, ι the identity, t_Z the canonical morphism $t_{[f]^*_Z}$ of (1.1), while bc^*_g and $\mathrm{bc}^!_g$ are the usual transitivity morphisms. Again, functoriality of Q follows from the fact that the transitivity morphisms and (1.1) are compatible with composition.

Now we have seen how **cor**′ "generalizes" the contexts discussed at the end of the previous paragraph, and the natural thing to do would be to prove functoriality of the trace map with respect to **cor**′. However, the conditions imposed on the morphisms in **cor**′ are too weak for that. We will thus pick out a class of morphisms ("cohomological morphisms") for which functoriality can be proved. Of course we then have to make sure the "functors" P, Q and S all land in this restricted class of morphisms.

Let $(\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ be a cohomological premorphism from c to $\overline{c}$. The following axioms will be of interest in the sequel:

(M1) Each t_Z is commutative and associative, i. e. the following two diagrams commute for every $Z \in Ob(c)$ and every $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C} \in \mathfrak{D}^b_{ctf}(Z)$:

$$\begin{array}{ccc} f\mathscr{A} \otimes f\mathscr{B} & \xrightarrow{t} & f(\mathscr{A} \otimes \mathscr{B}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ f\mathscr{B} \otimes f\mathscr{A} & \xrightarrow[t]{} & f(\mathscr{B} \otimes \mathscr{A}) \end{array} \qquad \begin{array}{ccc} f\mathscr{A} \otimes f\mathscr{B} \otimes f\mathscr{C} & \xrightarrow{t} & f\mathscr{A} \otimes f(\mathscr{B} \otimes \mathscr{C}) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ f(\mathscr{A} \otimes \mathscr{B}) \otimes f\mathscr{C} & \xrightarrow[t]{} & f(\mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}) \end{array} \tag{3.3}$$

(M2) For $\mathscr{A} \in \mathfrak{D}^b_{ctf}(Z)$, $Z \in Ob(c)$, the composition

$$\varphi : f_Z\mathscr{A} \xrightarrow{\cong} f_Z\mathscr{A} \otimes \Lambda_{\overline{Z}} \xrightarrow{\iota_Z} f_Z\mathscr{A} \otimes f_Z\Lambda_Z \xrightarrow{t_Z} f_Z(\mathscr{A} \otimes \Lambda_Z) \xrightarrow{\cong} f_Z\mathscr{A}$$

is the identity morphism.

(M3) For every $Z \in Ob(c)$, $\mathrm{bc}^*_{\mathbb{1}_Z}$ is the identity morphism, and for every $g_1 : Z_1 \to Z_2$, $g_2 : Z_2 \to Z_3$ in $Mor(c)$, $\mathrm{bc}^*_{g_2g_1}$ decomposes as

$$\mathrm{bc}^*_{g_2g_1} : (\overline{g_2g_1})^* f_{Z_3} \xrightarrow{\cong} \overline{g}_1^*\overline{g}_2^* f_{Z_3} \xrightarrow{\mathrm{bc}^*_{g_2}} \overline{g}_1^* f_{Z_2} g_2^* \xrightarrow{\mathrm{bc}^*_{g_1}} f_{Z_1} g_1^* g_2^* \xrightarrow{\cong} f_{Z_1}(g_2g_1)^*.$$

(M4) For every $g : Z_1 \to Z_2$ in $Mor(c)$ and every $\mathscr{A}$, $\mathscr{B} \in \mathfrak{D}^b_{ctf}(Z_2)$, the following diagram commutes:

$$\begin{array}{ccccc} \overline{g}^* f\mathscr{A} \otimes \overline{g}^* f\mathscr{B} & \xrightarrow[\cong]{t_{\overline{g}^*}} & \overline{g}^*(f\mathscr{A} \otimes f\mathscr{B}) & \xrightarrow{t} & \overline{g}^* f(\mathscr{A} \otimes \mathscr{B}) \\ \downarrow{\scriptstyle \mathrm{bc}^*_g \otimes \mathrm{bc}^*_g} & & & & \downarrow{\scriptstyle \mathrm{bc}^*_g} \\ f g^*\mathscr{A} \otimes f g^*\mathscr{B} & \xrightarrow[t]{} & f(g^*\mathscr{A} \otimes g^*\mathscr{B}) & \xrightarrow[\cong]{t_{g^*}} & f g^*(\mathscr{A} \otimes \mathscr{B}) \end{array} \tag{3.4}$$

(M5) For every $Z \in Ob(c)$, d_Z is an isomorphism.

(M6) For every $g \in Mor(c)$ with g and $\overline{g}$ proper, both bc_{*g} and $\mathrm{bc}_{!g}$ are isomorphisms, and inverses to each other.

(M7) For every $Z \in Ob(c)$, $\mathrm{bc}^!_{\mathbb{1}_Z}$ is the identity morphism, and for every $g_1 : Z_1 \to Z_2$, $g_2 : Z_2 \to Z_3$ in $Mor(c)$, $\mathrm{bc}^!_{g_2g_1}$ decomposes as

$$\mathrm{bc}^!_{g_2g_1} : (\overline{g_2g_1})^! f_{Z_3} \xrightarrow{\cong} \overline{g}_1^!\overline{g}_2^! f_{Z_3} \xrightarrow{\mathrm{bc}^!_{g_2}} \overline{g}_1^! f_{Z_2} g_2^! \xrightarrow{\mathrm{bc}^!_{g_1}} f_{Z_1} g_1^! g_2^! \xrightarrow{\cong} f_{Z_1}(g_2g_1)^!.$$

(M8) Let $g_4g_1 = g_2g_3$ be a universally cartesian square in $Mor(c)$. Then the following diagram commutes:

$$\begin{array}{ccc} \overline{g}_2^*\overline{g}_{4!} f & \xrightarrow{\mathrm{bc}\ (1.13)} & \overline{g}_{3!}\overline{g}_1^* f \\ \downarrow{\scriptstyle \mathrm{bc}^*_{g_2} \circ \mathrm{bc}_{!g_4}} & & \downarrow{\scriptstyle \mathrm{bc}_{!g_3} \circ \mathrm{bc}^*_{g_1}} \\ f g_2^* g_{4!} & \xrightarrow[\mathrm{bc}\ (1.13)]{} & f g_{3!} g_1^* \end{array} \tag{3.5}$$

(M9) Let $g : Z_1 \to Z_2$ be in $Mor(c)$. Then the following diagram commutes for any $\mathscr{A} \in \mathfrak{D}^b_{ctf}(Z_1)$, $\mathscr{B} \in \mathfrak{D}^b_{ctf}(Z_2)$:

$$\begin{array}{ccc} \overline{g}_!(f\mathscr{A}\otimes\overline{g}^*f\mathscr{B}) & \xrightarrow{t_{Z_1}\circ \mathrm{bc}_g^*} & \overline{g}_!f(\mathscr{A}\otimes g^*\mathscr{B}) \\ \uparrow{\scriptstyle \mathrm{proj}\ (1.4)} & & \downarrow{\scriptstyle \mathrm{bc}_{!g}} \\ \overline{g}_!f\mathscr{A}\otimes f\mathscr{B} & & fg_!(\mathscr{A}\otimes g^*\mathscr{B}) \\ \downarrow{\scriptstyle \mathrm{bc}_{!g}} & & \uparrow{\scriptstyle\mathrm{proj}\ (1.4)} \\ fg_!\mathscr{A}\otimes f\mathscr{B} & \xrightarrow[t_{Z_2}]{} & f(g_!\mathscr{A}\otimes\mathscr{B}) \end{array} \tag{3.6}$$

Example 3.7 Let us check which of these axioms are satisfied by the cohomological premorphisms considered in the previous example. More specifically, we will prove that any cohomological premorphism in the image of S or P satisfies Axioms (M1–6), while any cohomological premorphism in the image of Q satisfies Axioms (M1–5, 7–9).

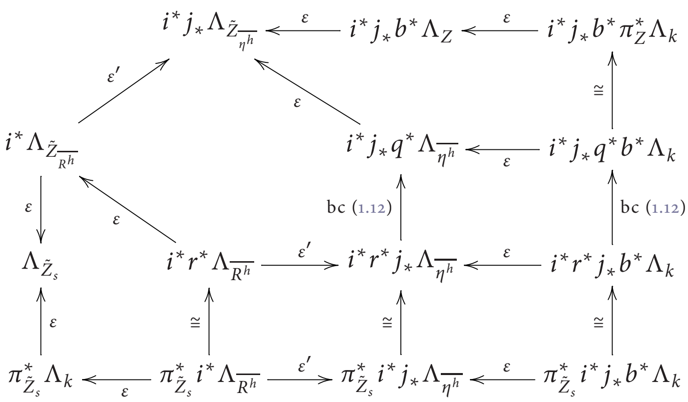
(M1) In the restriction case, the two squares in (3.3) commute since they do so already on the level of complexes. This is true for any morphism of schemes (not only for open immersions) hence the proper case follows by adjointness. Finally, the specialization case follows, in view of the definition of t_Z, from the two cases just considered.

(M2) Let $f : X \to Y$ be an arbitrary morphism of schemes. It has been remarked in 1.(i) that for any $\mathscr{A} \in \mathfrak{D}^b_{ctf}(X)$ and $\mathscr{B} \in \mathfrak{D}^b_{ctf}(Y)$, the following two squares commute:

$$\begin{array}{ccc} f^*\mathscr{B} & \xleftarrow{\cong} & f^*\mathscr{B}\otimes\Lambda_X \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong\ \varepsilon} \\ f^*(\mathscr{B}\otimes\Lambda_Y) & \xleftarrow[t_{f^*}]{\cong} & f^*\mathscr{B}\otimes f^*\Lambda_Y \end{array} \qquad \begin{array}{ccc} f_*\mathscr{A} & \xleftarrow{\cong} & f_*\mathscr{A}\otimes\Lambda \\ \uparrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\varepsilon'} \\ f_*(\mathscr{A}\otimes\Lambda) & \xleftarrow[t_{f_*}]{} & f_*\mathscr{A}\otimes f_*\Lambda \end{array} \tag{3.7}$$

Now, it follows from (1.7) that in the restriction case $\iota_Z = \varepsilon^{-1}$ hence the commutativity of the left diagram in (3.7) implies the axiom in this case. Similarly, $\iota_Z = \varepsilon'$ in the proper case hence the commutativity of the right square in (3.7) implies the axiom in this case.

For the specialization case we take up the notation of 3.6. In addition let $q : \tilde{Z}_{\overline{\eta^h}} \to \overline{\eta^h}$ and $r : \tilde{Z}_{\overline{R^h}} \to \overline{R^h}$ denote the canonical morphisms. Consider then the following diagram:



The commutativity of the two trapezoids follows immediately from the commutativity of (1.7), while the commutativity of the three squares in the lower right part of the diagram is clear by the naturality of the morphisms involved. Replacing the base change morphism (1.12) by

$$r^* j_* \xrightarrow{\text{adj}} r^* j_* q_* q^* \xrightarrow{\cong} r^* r_* j_* q^* \xrightarrow{\text{adj}} j_* q^*,$$

and ε' by $\Lambda \xrightarrow{\text{adj}} j_* j^* \Lambda \xrightarrow{\varepsilon} j_* \Lambda$, the commutativity of the pentagon becomes obvious. Notice that this diagram provides us with an alternative description of $\iota_Z : \Lambda_{\tilde{Z}_s} \to \mathrm{sp}_{\tilde{Z}} \Lambda_Z$ (cf. (3.2)). We use it in the following diagram (it appears as the leftmost column):

$$\begin{array}{ccccccc}
\mathrm{sp}_{\tilde Z}\mathscr{A} \otimes i^* j_* b^* \Lambda & \xrightarrow{t_{i*}} & i^*(j_* b^* \mathscr{A} \otimes j_* b^* \Lambda) & \xrightarrow{t_{j*}} & i^* j_* (b^* \mathscr{A} \otimes b^* \Lambda) & \xrightarrow{t_{b*}} & i^* j_* b^* (\mathscr{A} \otimes \Lambda) \\
\downarrow {\scriptstyle \mathbb{1}\otimes\varepsilon} & & \downarrow {\scriptstyle \mathbb{1}\otimes\varepsilon} & & \downarrow {\scriptstyle \mathbb{1}\otimes\varepsilon} & & \downarrow {\scriptstyle \cong} \\
\mathrm{sp}_{\tilde Z}\mathscr{A} \otimes i^* j_* \Lambda & \xrightarrow{t_{i*}} & i^*(j_* b^* \mathscr{A} \otimes j_* \Lambda) & \xrightarrow{t_{j*}} & i^* j_* (b^* \mathscr{A} \otimes \Lambda) & \xrightarrow{\cong} & \mathrm{sp}_{\tilde Z}\mathscr{A} \\
\uparrow {\scriptstyle \mathbb{1}\otimes\varepsilon'} & & \uparrow {\scriptstyle \mathbb{1}\otimes\varepsilon'} & & & \nearrow {\scriptstyle =} & \\
\mathrm{sp}_{\tilde Z}\mathscr{A} \otimes i^* \Lambda & \xrightarrow{t_{i*}} & i^*(j_* b^* \mathscr{A} \otimes \Lambda) & & & & \\
\downarrow {\scriptstyle \mathbb{1}\otimes\varepsilon} & & \downarrow {\scriptstyle \cong} & & & & \\
\mathrm{sp}_{\tilde Z}\mathscr{A} \otimes \Lambda & \xrightarrow{\cong} & \mathrm{sp}_{\tilde Z}\mathscr{A} & & & &
\end{array}$$

The bottom left square and the top right square are instances of the left, while the triangle is an instance of the right diagram in (3.7). The other squares are clearly commutative hence so is the whole diagram. By what we said above, the composition of the leftmost column and the top row is $t_Z \circ \iota_Z$ hence the axiom holds in the specialization case as well.

(M3) The first statement is clear. For the second one, see 1.**(k)** in the proper and the restriction case and 2.19.2 in the specialization case.

(M4) In the restriction case, the diagram commutes already on the level of complexes, and (this again not being specific to open immersions) the proper case follows from this one by adjointness. Finally, the specialization case follows from these two.

(M5) To prove that d_Z is an isomorphism, we claim, in the proper case, that it coincides with the local Verdier duality isomorphism

$$f_* \,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, f^! K) \xrightarrow{\cong} \mathbf{R}\underline{\mathrm{Hom}}(f_! \mathscr{A}, K)$$

(see [2, XVIII, 3.1.10]). Indeed, the latter is defined as the composition

$$f_* \,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, f^! K) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}(f_! \mathscr{A}, f_! f^! K) \xrightarrow{\text{adj}} \mathbf{R}\underline{\mathrm{Hom}}(f_! \mathscr{A}, K),$$

where the first map is just r_Z. Hence the claim follows from the fact that $\tilde{\pi}_Z$ coincides with the morphism adj above, i. e. that the following diagram commutes (which is obvious):

$$\begin{array}{ccc}
f_! f^! \pi^! & \xrightarrow{\cong} & f_! \pi^! \\
\downarrow {\scriptstyle \text{adj}} & \swarrow {\scriptstyle \text{bc (1.11)}} & \\
\pi^! & &
\end{array}$$

In the restriction case, both r_Z and $\tilde{\pi}_Z$ are clearly isomorphisms hence so is d_Z.

In the specialization case, let $Z \in Ob(c)$ and let $(\tilde{Z}, \varphi)$ be its corresponding lift over R. Denote by b the composition

$$\tilde{Z}_{\eta^h} \longrightarrow \tilde{Z}_\eta \xrightarrow[\cong]{\varphi_\eta} Z_\eta \longrightarrow Z,$$

by p the canonical map $\tilde{Z}_{\eta^h} \to \eta^h$, and consider the following diagram ($\mathscr{F} \in \mathfrak{D}^b_{ctf}(Z)$ arbitrary, $[-,-]$ abbreviates $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$):

$$\begin{array}{ccccc}
\Psi b^*[\mathscr{F}, K_Z] \otimes \Psi b^* \mathscr{F} & \longrightarrow & \Psi(b^*[\mathscr{F}, K_Z] \otimes b^*\mathscr{F}) & \xrightarrow[\cong]{t_{b^*}} & \Psi b^*([\mathscr{F}, K_Z] \otimes \mathscr{F}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\
\Psi[b^*\mathscr{F}, b^*K_Z] \otimes \Psi b^*\mathscr{F} & \longrightarrow & \Psi([b^*\mathscr{F}, b^*K_Z] \otimes b^*\mathscr{F}) & \xrightarrow{\mathrm{ev}} & \Psi b^* K_Z \\
{\scriptstyle \mathrm{bc}\ (1.15)}\downarrow & & \downarrow{\scriptstyle \mathrm{bc}\ (1.15)} & & \downarrow{\scriptstyle \mathrm{bc}\ (1.15)} \\
\Psi[b^*\mathscr{F}, p^!\Lambda] \otimes \Psi b^*\mathscr{F} & \longrightarrow & \Psi([b^*\mathscr{F}, p^!\Lambda] \otimes b^*\mathscr{F}) & \xrightarrow[\mathrm{ev}]{} & \Psi p^!\Lambda
\end{array}$$

Here the horizontal arrows in the left part of the diagram are induced by t_{j_*} as in the definition of t_Z. In particular the top row is precisely t_Z. The unlabeled vertical arrows are induced by the canonical morphism $b^* \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, K_Z) \to \mathbf{R}\underline{\mathrm{Hom}}(b^*\mathscr{F}, b^*K_Z)$ adjoint to

$$b^* \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, K_Z) \otimes b^*\mathscr{F} \xrightarrow[\cong]{t_{b^*}} b^*(\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, K_Z) \otimes \mathscr{F}) \xrightarrow{\mathrm{ev}} b^* K_Z. \tag{3.8}$$

Hence the two upper squares are commutative. Finally, the vertical arrows in the lower half are all induced by

$$b^*\pi_Z^! \Lambda_k \to p^! c^* \Lambda_k \xrightarrow[\cong]{\varepsilon} p^! \Lambda_{\eta^h}, \tag{3.9}$$

where $c : \eta^h \to k$. Thus the lower half and hence the whole diagram is commutative.

Now, the composition of the top row t_Z and the right column followed by the base change morphism

$$\Psi p^! \Lambda_{\eta^h} \xrightarrow{(1.25)} \pi^!_{\tilde{Z}_s} \Psi \Lambda_{\eta^h} = \pi^!_{\tilde{Z}_s} i^* j_* \Lambda_{\overline{\eta^h}} \xrightarrow[\cong]{\varepsilon'^{-1}} \pi^!_{\tilde{Z}_s} i^* \Lambda_{\overline{R^h}} \xrightarrow[\cong]{\varepsilon} K_{\tilde{Z}_s}$$

(notation as in 1.§4) is adjoint to d_Z hence we have shown that d_Z factors as follows:

$$\Psi b^* \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{F}, K_Z) \to \Psi\, \mathbf{R}\underline{\mathrm{Hom}}(b^*\mathscr{F}, p^!\Lambda) \to \mathbf{R}\underline{\mathrm{Hom}}(\Psi b^*\mathscr{F}, K_{\tilde{Z}_s}).$$

The second map of this factorization is an isomorphism by [16, 4.2]. To prove that d_Z is an isomorphism we thus have to show that the first map in the factorization is. In fact, we will prove the stronger assertion that both (3.9) and the morphism adjoint to (3.8) are isomorphisms.

For this notice the following fact. Given any morphism $f : Y \to Z$ and the associated cartesian square

$$\begin{array}{ccc}
Y' & \xrightarrow{b'} & Y \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\tilde{Z}_{\eta^h} & \xrightarrow[b]{} & Z
\end{array} \tag{3.10}$$

with b as above, the base change morphism (1.12) : $b^* f_* \to f'_* b'^*$ is an isomorphism. Indeed, the field extension $\eta^h \supset k$ is the direct limit of its smooth k-subalgebras $(k_\mu)_\mu$. By base change with respect to π_Z and $\pi_Z f$ we deduce inverse systems of schemes $(Z_\mu)_\mu$ and $(Y_\mu)_\mu$ with morphisms denoted as in the diagram (of cartesian squares)

$$\begin{array}{ccccc} Y' & \xrightarrow{b'_{\infty\to\mu}} & Y_\mu & \xrightarrow{b'_\mu} & Y \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f_\mu}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \tilde{Z}_{\eta^h} & \xrightarrow[b_{\infty\to\mu}]{} & Z_\mu & \xrightarrow[b_\mu]{} & Z \end{array}$$

decomposing (3.10), $\tilde{Z}_{\eta^h}$ and Y' being the inverse limits, respectively. Consider the following diagram where the unlabeled arrows are the canonical morphisms:

$$\begin{array}{ccccc} \varinjlim_\mu b^*_{\infty\to\mu} f_{\mu*} b'^*_\mu & \xrightarrow{\text{bc (1.12)}} & \varinjlim_\mu f'_* b'^*_{\infty\to\mu} b'^*_\mu & \xrightarrow{\cong} & f'_* b'^* \\ {\scriptstyle \text{bc (1.12)}}\big\uparrow & & & & \big\uparrow{\scriptstyle \text{bc (1.12)}} \\ \varinjlim_\mu b^*_{\infty\to\mu} b^*_\mu f_* & & \xrightarrow[\cong]{} & & b^* f_* \end{array}$$

By the compatibility of (1.12) with respect to composition (cf. 1.(**k**)), this diagram commutes. By [1, VII, 5.11], the top row is an isomorphism, while the left vertical arrow is an isomorphism since b_μ is smooth. Hence so is the right vertical arrow.

Using the fact just established, the adjoint of (3.8) is shown to be an isomorphism exactly as in the proof of [14, I, 4.3]. For (3.9), we note that the claim is local on Z hence we may assume that π_Z factors into gh, h a closed immersion and g a smooth morphism. By the compatibility of the base change morphism (1.15) with respect to composition and since the claim is clear in the case of $\pi_Z = g$ smooth we are reduced to show that $h^!$ commutes with the pullback along $\eta^h \to k$. Let U be the complement of the image of $h : V \hookrightarrow X$ and let $f : U \hookrightarrow X$ be the associated open immersion. Furthermore, let's fix our notation as in the following diagram of cartesian squares:

$$\begin{array}{ccc} V' & \xrightarrow{e''} & V \\ {\scriptstyle h'}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ X' & \xrightarrow{e'} & X \\ {\scriptstyle f'}\big\uparrow & & \big\uparrow{\scriptstyle f} \\ U' & \xrightarrow{e} & U \end{array}$$

Here, e' denotes the base change of $\eta^h \to k$ with respect to π_X. Moreover, let $I^\cdot$ be a bounded below complex of injective sheaves on X, and let $\alpha : e'^* I^\cdot \to J^\cdot$ be an injective resolution of $e'^* I^\cdot$. Fix some $n \in \mathbb{Z}$. Applying the functor $\underline{\mathrm{Hom}}(-, I^n)$ to the canonical short exact sequence of sheaves on X

$$0 \longrightarrow f_! f^* \Lambda_X \xrightarrow{\text{adj}} \Lambda_X \xrightarrow{\text{adj}} h_* h^* \Lambda_X \longrightarrow 0$$

results in another short exact sequence of sheaves,

$$0 \longrightarrow h_* h^! I^n \xrightarrow{\text{adj}} I^n \xrightarrow{\text{adj}} f_* f^* I^n \longrightarrow 0.$$

Letting n vary and applying the exact functor h^* we obtain a short exact sequence of complexes

$$0 \longrightarrow h^! I^\cdot \xrightarrow{\beta} h^* I^\cdot \xrightarrow{\text{adj}} h^* f_* f^* I^\cdot \longrightarrow 0,$$

where β is induced by the composition

$$h^! \underset{\cong}{\xleftarrow{\text{adj}}} h^* h_* h^! = h^* h_! h^! \xrightarrow{\text{adj}} h^*.$$

Similar remarks apply to $J^\cdot$ and the morphisms h' and f'. We are now going to construct a morphism of distinguished triangles associated to such short exact sequences.

Still on the level of complexes, consider the following diagram:

$$\begin{array}{ccc} e''^* h^! I^\cdot & \xrightarrow{e''^* \beta_{I^\cdot}} & e''^* h^* I^\cdot \\ \downarrow{\scriptstyle \text{bc}} & & \downarrow{\scriptstyle \cong} \\ h'^! e'^* I^\cdot & \xrightarrow{\beta_{e'^* I^\cdot}} & h'^* e'^* I^\cdot \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ h'^! J^\cdot & \xrightarrow{\beta_{J^\cdot}} & h'^* J^\cdot \end{array} \tag{3.11}$$

Here, the base change morphism bc is induced by the composition

$$e''^* h^! \xrightarrow{\text{adj}} h'^! h'_! e''^* h^! \underset{\cong}{\xleftarrow{\text{adj}}} h'^! e'^* h_! h^! \xrightarrow{\text{adj}} h'^! e'^*,$$

the second arrow being the proper base change isomorphism (for sheaves). It is then easy to see that the upper half of (3.11) commutes while the lower half commutes by the naturality of β. Denote by γ (resp. δ) the composition of the vertical morphisms on the left (resp. right) of (3.11). We will define below an isomorphism $e''^* h^* j_* j^* I^\cdot \to h'^* j'_* j'^* J^\cdot$ in the derived category rendering commutative the following diagram:

$$\begin{array}{ccccccccc} e''^* h^! I^\cdot & \xrightarrow{e''^* \beta_{I^\cdot}} & e''^* h^* I^\cdot & \xrightarrow{\text{adj}} & e''^* h^* j_* j^* I^\cdot & \longleftarrow & C(e''^* \beta_{I^\cdot}) & \longrightarrow & e''^* h^* I^\cdot[1] \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \vdots{\scriptstyle \cong} & & \downarrow{\scriptstyle \delta \oplus \gamma[1]} & & \downarrow{\scriptstyle \gamma[1]} \\ h'^! J^\cdot & \xrightarrow[\beta_{J^\cdot}]{} & h'^* J^\cdot & \xrightarrow[\text{adj}]{} & h'^* j'_* j'^* J^\cdot & \longleftarrow & C(\beta_{J^\cdot}) & \longrightarrow & h^* J^\cdot[1] \end{array}$$

Here $C(f)$ denotes the mapping cone of f and the unlabeled morphisms are the canonical ones. Thus we obtain a morphism of distinguished triangles in the derived category. Since δ is an isomorphism we conclude that γ is an isomorphism too. But modulo the identification of the functor $h^!$ (on complexes of sheaves) and $h^!$ (its derived counterpart) when applied to complexes of injective sheaves, γ is just bc (1.15). Applying this to the case where $I^\cdot$ is an injective resolution of $g^! \Lambda_k$, we see that (3.9) must be an isomorphism.

As for the dotted arrow above we may choose the following composition of morphisms in the derived category (again, modulo the identification of j_* (on complexes of sheaves) and its derived counterpart when applied to complexes of injective sheaves):

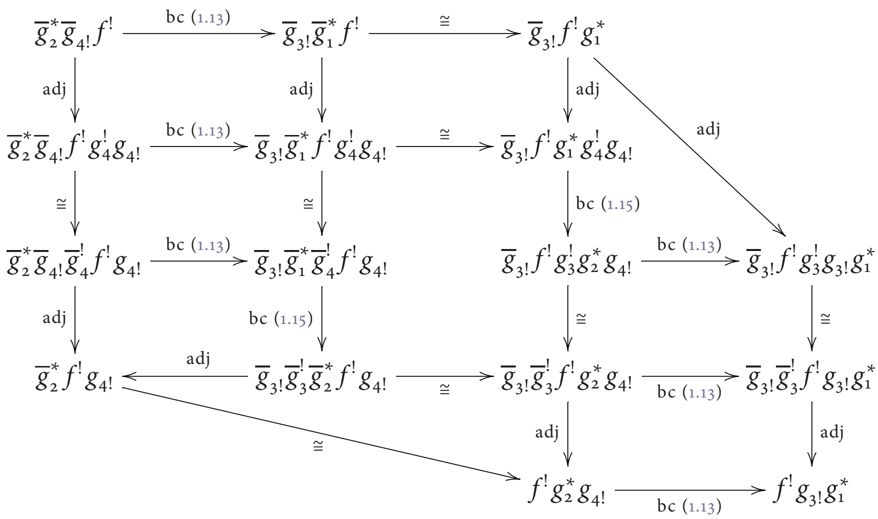
$$e''^* h^* j_* j^* I^\cdot \xrightarrow{\cong} h'^* e'^* j_* j^* I^\cdot \xrightarrow{\text{bc (1.12)}} h'^* j'_* e^* j^* I^\cdot \xrightarrow{\cong} h'^* j'_* j'^* e'^* I^\cdot \underset{\cong}{\xrightarrow{\alpha}} h'^* j'_* j'^* J^\cdot.$$

It is easily seen to make the above diagram commutative. Moreover, by the fact established above, the second arrow in the composition is an isomorphism, thus the claim.

(M6) Notice that in the proper case bc_{*g} and $\mathrm{bc}_{!g}$ are the usual transitivity morphisms $[f]_{Z_2*}g_* \to \overline{g}_*[f]_{Z_1*}$ and $\overline{g}_![f]_{Z_1!} \to [f]_{Z_2!}g_!$ hence the axiom is clearly satisfied.
The specialization case was treated in 2.19.1.

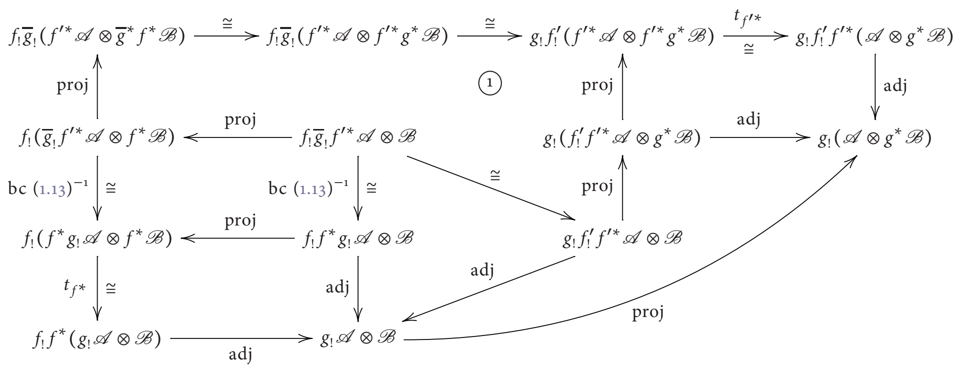
(M7) This is obvious in the restriction case (1.(k)).

(M8) In the restriction case, the diagram may be expanded as follows, where all unlabeled arrows are transitivity isomorphisms:



The big square in the middle of the diagram is easily seen to commute by the compatibility of (1.15) with respect to composition, the rest of the diagram is clearly commutative.

(M9) In the restriction case we are given a commutative square $f\overline{g} = gf'$ with f and f' open immersions. Then the diagram (3.6) extends by adjointness as follows:



① commutes by the compatibility of proj with respect to composition. The commutativity of the bottom left and the top right inner square can be checked on the level of complexes where it is obvious. The rest of the diagram clearly commutes.

Definition 3.8 1. Let c° and $\overline{c}^{\circ}$ be open subcorrespondences of c and $\overline{c}$, respectively. Denote the canonical functors $\mathbf{D}(c) \to \mathbf{D}(c^{\circ})$ and $\mathbf{D}(\overline{c}) \to \mathbf{D}(\overline{c}^{\circ})$ by $(\cdot)^{\circ}$. We say that a cohomological premorphism $(\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\})$ from c to $\overline{c}$ *extends* a cohomological premorphism $(\{f_{Z^{\circ}}, t_{Z^{\circ}}\}, \{\mathrm{bc}^*_{g^{\circ}}, \mathrm{bc}^!_{g^{\circ}}\})$ from c° to $\overline{c}^{\circ}$ if it comes equipped, for each $Z \in Ob(c)$ and each $\mathscr{A} \in \mathfrak{D}^b_{ctf}(Z)$, with an isomorphism

$$\chi_{Z,\mathscr{A}} : f_{Z^{\circ}}(\mathscr{A}|_{Z^{\circ}}) \xrightarrow{\cong} f_Z(\mathscr{A})|_{\overline{Z}^{\circ}},$$

natural in $\mathscr{A}$, such that the following three diagrams commute (for any Z, $\mathscr{A}$, $\mathscr{B}$, and $g : Z_1 \to Z_2$):

$$\begin{array}{ccccc} f_{Z^{\circ}}(\mathscr{A}|_{Z^{\circ}}) \otimes f_{Z^{\circ}}(\mathscr{B}|_{Z^{\circ}}) & \xrightarrow[\chi]{\cong} & f_Z(\mathscr{A})|_{\overline{Z}^{\circ}} \otimes f_Z(\mathscr{B})|_{\overline{Z}} & \xrightarrow{\cong} & (f_Z(\mathscr{A}) \otimes f_Z(\mathscr{B}))|_{\overline{Z}^{\circ}} \\ \downarrow{\scriptstyle t_{Z^{\circ}}} & & & & \downarrow{\scriptstyle t_Z} \\ f_{Z^{\circ}}(\mathscr{A}|_{Z^{\circ}} \otimes \mathscr{B}|_{Z^{\circ}}) & \xrightarrow{\cong} & f_{Z^{\circ}}((\mathscr{A} \otimes \mathscr{B})|_{Z^{\circ}}) & \xrightarrow[\chi]{\cong} & f_Z(\mathscr{A} \otimes \mathscr{B})|_{\overline{Z}^{\circ}} \end{array} \quad (3.12a)$$

$$\begin{array}{ccccc} \overline{g}^{\circ *} f_{Z_2^{\circ}}(\mathscr{A}|_{Z_2^{\circ}}) & \xrightarrow[\chi]{\cong} & \overline{g}^{\circ *}(f_{Z_2}(\mathscr{A})|_{\overline{Z}_2^{\circ}}) & \xrightarrow{\cong} & (\overline{g}^* f_{Z_2} \mathscr{A})|_{\overline{Z}_1^{\circ}} \\ \downarrow{\scriptstyle \mathrm{bc}^*_{g^{\circ}}} & & & & \downarrow{\scriptstyle \mathrm{bc}^*_g} \\ f_{Z_1^{\circ}} g^{\circ *}(\mathscr{A}|_{Z_2^{\circ}}) & \xrightarrow{\cong} & f_{Z_1^{\circ}}(g^*(\mathscr{A})|_{Z_1^{\circ}}) & \xrightarrow[\chi]{\cong} & (f_{Z_1} g^* \mathscr{A})|_{\overline{Z}_1^{\circ}} \end{array} \quad (3.12b)$$

$$\begin{array}{ccccc} (f_{Z_1} g^! \mathscr{A})|_{\overline{Z}_1^{\circ}} & \xleftarrow[\chi]{\cong} & f_{Z_1^{\circ}}((g^! \mathscr{A})|_{Z_1^{\circ}}) & \xrightarrow{\mathrm{bc}\ (1.15)} & f_{Z_1^{\circ}} g^{\circ !}(\mathscr{A}|_{Z_2^{\circ}}) \\ \downarrow{\scriptstyle \mathrm{bc}^!_g} & & & & \downarrow{\scriptstyle \mathrm{bc}^!_{g^{\circ}}} \\ (\overline{g}^! f_{Z_2} \mathscr{A})|_{\overline{Z}_1^{\circ}} & \xrightarrow{\mathrm{bc}\ (1.15)} & \overline{g}^!((f_{Z_2} \mathscr{A})|_{\overline{Z}_2^{\circ}}) & \xleftarrow[\chi]{\cong} & \overline{g}^{\circ !} f_{Z_2^{\circ}}(\mathscr{A}|_{Z_2^{\circ}}) \end{array} \quad (3.12c)$$

In other words, we require that this isomorphism identifies $t_{Z^{\circ}}$, $\mathrm{bc}^*_{g^{\circ}}$ and $\mathrm{bc}^!_{g^{\circ}}$ with $t_Z|_{\overline{Z}^{\circ}}$, $\mathrm{bc}^*_g|_{\overline{Z}_1^{\circ}}$ and $\mathrm{bc}^!_g|_{\overline{Z}_1^{\circ}}$, respectively.

2. A cohomological premorphism f from c to $\overline{c}$ is called a *good cohomological morphism (of type (C1), (C2)*, respectively) if the following condition holds, respectively:

   (C1) f satisfies Axioms (M1–6) above and it can be extended to a cohomological premorphism between compactifications of c and $\overline{c}$ which also satisfies Axioms (M1–6).

   (C2) f satisfies Axioms (M1–5, 7–9).

3. A *cohomological morphism* is a cohomological premorphism which decomposes into finitely many good cohomological morphisms.

Clearly, the self-correspondences together with the cohomological morphisms define a subcategory of $\mathbf{cor}'$, denoted $\mathbf{cor}$.

Example 3.9 Our next task is to check that the functors S, Q and P factor through $\mathbf{cor} \hookrightarrow \mathbf{cor}'$. We have already proved in the previous example that any morphism in the image of Q is actually a good cohomological morphism (of type (C2)).

For P, let $[f] : c \to b$ be a proper morphism of self-correspondences. By 2.5.2, there exist compactifications $[i] : c \hookrightarrow \overline{c}$ and $[j] : b \hookrightarrow \overline{b}$ and a morphism $[\overline{f}] : \overline{c} \to \overline{b}$ extending $[f]$. We have proved in the previous example that both $P([f])$ and $P([\overline{f}])$ satisfy Axioms (M1–6). To prove that $P([f])$ is a good cohomological morphism (of type (C1)) it thus suffices to show that $P([\overline{f}])$ extends $P([f])$. If Z is in $Ob(c)$ and if Z' is the corresponding element of $Ob(b)$ then, in the notation of 3.6, there is a canonical identification of functors $[f]_{Z*}[i]^*_{\overline{Z}} \cong [j]^*_{\overline{Z}'}[\overline{f}]_{\overline{Z}*}$ coming from the natural identification of the corresponding functors of sheaves since the restriction functors are exact and preserve injectives. It is then clear that this isomorphism of functors satisfies the properties needed for $P([\overline{f}])$ to extend $P([f])$.

We now turn to S, the specialization functor. Let c be a correspondence and let $\tilde{c}$ be a lift of c over R, defining a morphism $c \to \tilde{c}_s$ in **SCor**. By 2.5.1, there exists a compactification $c \hookrightarrow \overline{c}$ of c. Let $\tilde{d}$ and $\tilde{d}(\overline{c})$ be correspondences over R as in 2.27, thus $[f] : \tilde{d} \to \tilde{c}$ is proper, $\tilde{d}$ lifts c, and embeds as an open immersion into the correspondence $\tilde{d}(\overline{c})$ which is proper over R and lifts $\overline{c}$. Moreover, the diagram (2.12) commutes. We then arrive at the situation depicted in the following diagram in **cor′**:

$$\begin{array}{ccc} c & \xrightarrow{S(\tilde{d})} & \tilde{d}_s \\ & {\scriptstyle S(\tilde{c})}\searrow \quad \swarrow{\scriptstyle P([f_s])} & \\ & \tilde{c}_s & \\ & & \\ \overline{c} & \xrightarrow{S(\tilde{d}(\overline{c}))} & \tilde{d}(\overline{c})_s \end{array} \tag{3.13}$$

(Here, the components of $[f_s]$ arise from the components of $[f]$ by the base change $k \to R$.) Assume for the moment that the triangle commutes, i. e. assume that we have a factorization of $S(\tilde{c})$ as $P([f_s]) \circ S(\tilde{d})$. By what we showed above, $P([f_s])$ is a good cohomological morphism. Moreover, the diagram already suggests a candidate, $S(\tilde{d}(\overline{c}))$, for extending $S(\tilde{d})$ to a compactification. Since we have proved in the previous example that all premorphisms in the image of S satisfy Axioms (M1–6), this would show that $S(\tilde{d})$ is a good cohomological morphism as well, and thus imply that $S(\tilde{c})$ indeed is a morphism of **cor**.

It thus remains to show the commutativity of the diagram and the fact that the arrow at the bottom, $S(\tilde{d}(\overline{c}))$, extends the arrow at the top, $S(\tilde{d})$. Let us check the latter first. Clearly, c is an open subcorrespondence of $\overline{c}$, and $\tilde{d}_s$ is an open subcorrespondence of $\tilde{d}(\overline{c})_s$. Taking up the notation of 3.6, let $Z \in Ob(c)$, and denote by $j : Z \hookrightarrow \overline{Z}$ and $\tilde{j} : \tilde{Z} \to \tilde{Z}'$ the open immersions induced by $c \hookrightarrow \overline{c}$ and $\tilde{d} \hookrightarrow \tilde{d}(\overline{c})$, respectively. Then, the commutativity of (2.12) implies that $\tilde{j}$ lifts j. Hence there is a base change morphism

$$\mathrm{sp}_{\tilde{Z}} j^! \xrightarrow{\mathrm{bc}^!} \tilde{j}^!_s \mathrm{sp}_{\tilde{Z}'},$$

which is an isomorphism (2.19.1). We take this as χ (in the notation of 3.8). We still have to prove that the diagrams (3.12) in the definition commute. For this note that $\chi = \mathrm{bc}^!$ is the inverse of bc^* (again, by 2.19.1). Thus the diagrams (3.12b) and (3.12c) simply express the compatibility of the base change morphisms with respect to composition which has already been proved (2.19.2). Finally, the diagram (3.12a) is equal to the diagram (3.4) of Axiom (M4) and is thus commutative. (In the proof of this axiom in the previous example we didn't use the fact that $g : Z_1 \to Z_2$ is an element of $Mor(c)$.)

We now turn to the commutativity of the triangle in (3.13). Let $Z \in Ob(c)$ and denote by $\tilde{Z}$ and $\tilde{Z}'$ its lifts defined by $\tilde{c}$ and $\tilde{d}$, respectively. Also, denote by $\tilde{f} : \tilde{Z}' \to \tilde{Z}$ the morphism defined by $[f] : \tilde{d} \to \tilde{c}$. By the choice of $\tilde{d}$, $\tilde{f}$ is a lift of the identity morphism $\mathbb{1}_Z$. Hence there is a base change morphism

$$\mathrm{sp}_{\tilde{Z}} = \mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*} \xrightarrow{\mathrm{bc}_*} \tilde{f}_{s*} \mathrm{sp}_{\tilde{Z}'},$$

which is an isomorphism since $\tilde{f}$ is proper (2.19.1). This provides an identification of the functors $\mathfrak{D}^b_{ctf}(Z) \to \mathfrak{D}^b_{ctf}(\tilde{Z}_s)$ defined by $S(\tilde{c})$ and $P([f_s]) \circ S(\tilde{d})$. We have to check that this identification is "compatible" with the other data making up these two cohomological premorphisms, i. e. that it identifies the morphisms t_Z, bc^*_g, $\mathrm{bc}^!_g$, ι_Z of the two premorphisms ($Z \in Ob(c)$, $g \in Mor(c)$). Let us check this first with respect to t_Z. Here, the compatibility means that the following diagram commutes for any $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(Z)$:

$$\begin{array}{ccccc}
\mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}\mathscr{A} \otimes \mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}\mathscr{B} & \xrightarrow{t_{\tilde{Z}}} & \mathrm{sp}_{\tilde{Z}}(\mathbb{1}_{Z*}\mathscr{A} \otimes \mathbb{1}_{Z*}\mathscr{B}) & \xrightarrow[=]{t_{\mathbb{1}_{Z*}}} & \mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}(\mathscr{A} \otimes \mathscr{B}) \\
\downarrow{\scriptstyle \mathrm{bc}_*} & & & & \downarrow{\scriptstyle \mathrm{bc}_*} \\
\tilde{f}_{s*}\mathrm{sp}_{\tilde{Z}'}\mathscr{A} \otimes \tilde{f}_{s*}\mathrm{sp}_{\tilde{Z}'}\mathscr{B} & \xrightarrow[t_{\tilde{f}_{s*}}]{} & \tilde{f}_{s*}(\mathrm{sp}_{\tilde{Z}'}\mathscr{A} \otimes \mathrm{sp}_{\tilde{Z}'}\mathscr{B}) & \xrightarrow[t_{\tilde{Z}'}]{} & \tilde{f}_{s*}\mathrm{sp}_{\tilde{Z}'}(\mathscr{A} \otimes \mathscr{B})
\end{array}$$

By adjointness, this diagram translates to the following one:

$$\begin{array}{ccccc}
\tilde{f}^*_s(\mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}\mathscr{A} \otimes \mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}\mathscr{B}) & \xrightarrow{t_{\tilde{Z}}} & \tilde{f}^*_s \mathrm{sp}_{\tilde{Z}}(\mathbb{1}_{Z*}\mathscr{A} \otimes \mathbb{1}_{Z*}\mathscr{B}) & \xrightarrow[=]{t_{\mathbb{1}_{Z*}}} & \tilde{f}^*_s \mathrm{sp}_{\tilde{Z}} \mathbb{1}_{Z*}(\mathscr{A} \otimes \mathscr{B}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{bc}^*} & & \downarrow{\scriptstyle \mathrm{bc}^*} \\
\tilde{f}_{s*}\mathrm{sp}_{\tilde{Z}'} \mathbb{1}_{Z*}\mathscr{A} \otimes \tilde{f}_{s*}\mathrm{sp}_{\tilde{Z}'} \mathbb{1}_{Z*}\mathscr{B} & & \mathrm{sp}_{\tilde{Z}'} \mathbb{1}^*_Z(\mathbb{1}_{Z*}\mathscr{A} \otimes \mathbb{1}_{Z*}\mathscr{B}) & \xrightarrow[=]{t_{\mathbb{1}_{Z*}}} & \mathrm{sp}_{\tilde{Z}'} \mathbb{1}^*_Z \mathbb{1}_{Z*}(\mathscr{A} \otimes \mathscr{B}) \\
\downarrow{\scriptstyle \mathrm{bc}^*} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \mathrm{adj}} \\
\mathrm{sp}_{\tilde{Z}'} \mathbb{1}^*_Z \mathbb{1}_{Z*}\mathscr{A} \otimes \mathrm{sp}_{\tilde{Z}'} \mathbb{1}^*_Z \mathbb{1}_{Z*}\mathscr{B} & \xrightarrow[t_{\tilde{Z}'}]{} & \mathrm{sp}_{\tilde{Z}'}(\mathbb{1}^*_Z \mathbb{1}_{Z*}\mathscr{A} \otimes \mathbb{1}^*_Z \mathbb{1}_{Z*}\mathscr{B}) & \xrightarrow[\mathrm{adj}]{} & \mathrm{sp}_{\tilde{Z}'}(\mathscr{A} \otimes \mathscr{B})
\end{array}$$

The right upper square commutes by the naturality of bc^*, the right lower square clearly commutes, while the left half of the diagram is again the same as (3.4) of Axiom (M4), thus is commutative for the same reasons.

In the case of the base change morphisms which are part of the data making up the premorphisms $S(\tilde{c})$ and $P([f]) \circ S(\tilde{d})$, the compatibility is easy to check. E. g. in the case of bc^*_g, $g : Z_1 \to Z_2$ in $Mor(c)$:

$$\begin{array}{ccccc}
\tilde{g}^*_s \mathrm{sp}_{\tilde{Z}_2} \mathbb{1}_{Z_2*} & \xrightarrow{\mathrm{bc}^*_g} & \mathrm{sp}_{\tilde{Z}_1} g^* \mathbb{1}_{Z_2*} & \xrightarrow[=]{\mathrm{bc}\ (1.12)} & \mathrm{sp}_{\tilde{Z}_1} \mathbb{1}_{Z_1*} g^* \\
\downarrow{\scriptstyle \mathrm{bc}_*} & & & & \downarrow{\scriptstyle \mathrm{bc}_*} \\
\tilde{g}^*_s \tilde{f}_{2s*} \mathrm{sp}_{\tilde{Z}'_2} & \xrightarrow[\mathrm{bc}\ (1.12)]{} & \tilde{f}_{1s*} \tilde{g}'^*_s \mathrm{sp}_{\tilde{Z}'_2} & \xrightarrow[\mathrm{bc}^*_g]{} & \tilde{f}_{1s*} \mathrm{sp}_{\tilde{Z}_1} g^*
\end{array}$$

(Here, $\tilde{g} : \tilde{Z}_1 \to \tilde{Z}_2$, $\tilde{g}' : \tilde{Z}'_1 \to \tilde{Z}'_2$, $\tilde{f}_i : \tilde{Z}'_i \to \tilde{Z}_i$, $i = 1, 2$.) The commutativity of this diagram follows from the description of bc_* in terms of bc^* (by adjointness), and from the compatibility of bc^* with respect to composition (2.19.2). The case of $\mathrm{bc}^!_g$ is treated in the same way, using the fact that $\mathrm{bc}_* = \mathrm{bc}_!^{-1}$ (2.19.1).

Finally, $\iota : \mathrm{sp}_R \Lambda_k \to \Lambda_k$ is exactly the same for both cohomological premorphisms $S(\tilde{c})$ and $P([f]) \circ S(\tilde{d})$, while the identifying base-change morphism $\mathrm{sp}_R \mathbb{1}_* \Lambda_R \to \mathbb{1}_* \mathrm{sp}_R$ is the identity. This concludes the proof of the fact that the functor S also factors through $\mathbf{cor} \hookrightarrow \mathbf{cor}'$.

§3 Properties of good cohomological morphisms

Throughout this paragraph we fix a good cohomological morphism $F = (\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ from $c : C \to X \times X$ to $\overline{c}$.

Lemma 3.10 1. *F satisfies Axiom (M7).*

2. *For each $g : Z_1 \to Z_2$ in Mor(c), the morphism $\tilde{\pi}_{Z_1}$ admits a decomposition*

$$f_{Z_1} K_{Z_1} \xrightarrow{\cong} f_{Z_1} g^! K_{Z_2} \xrightarrow{\mathrm{bc}^!_g} \overline{g}^! f_{Z_2} K_{Z_2} \xrightarrow{\tilde{\pi}_{Z_2}} \overline{g}^! K_{\overline{Z}_2} \xrightarrow{\cong} K_{\overline{Z}_2}.$$

3. *For each commutative square $g_4 g_1 = g_2 g_3$ in Mor(c), the following two diagrams commute:*

$$\begin{array}{ccccc} f g_2^* g_{4*} & \xleftarrow{\mathrm{bc}^*_{g_2}} & \overline{g}_2^* f g_{4*} & \xrightarrow{\mathrm{bc}_{*g_4}} & \overline{g}_2^* \overline{g}_{4*} f \\ \downarrow{\scriptstyle \mathrm{bc}\ (1.12)} & & & & \downarrow{\scriptstyle \mathrm{bc}\ (1.12)} \\ f g_{3*} g_1^* & \xrightarrow[\mathrm{bc}_{*g_3}]{} & \overline{g}_{3*} f g_1^* & \xleftarrow[\mathrm{bc}^*_{g_1}]{} & \overline{g}_{3*} \overline{g}_1^* f \end{array} \tag{3.14a}$$

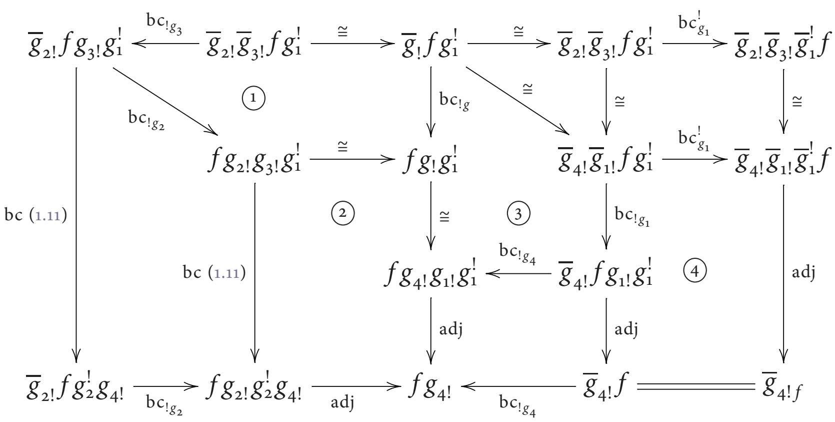
$$\begin{array}{ccccc} f g_{3!} g_1^! & \xleftarrow{\mathrm{bc}_{!g_3}} & \overline{g}_{3!} f g_1^! & \xrightarrow{\mathrm{bc}^!_{g_1}} & \overline{g}_{3!} \overline{g}_1^! f \\ \downarrow{\scriptstyle \mathrm{bc}\ (1.11)} & & & & \downarrow{\scriptstyle \mathrm{bc}\ (1.11)} \\ f g_2^! g_{4!} & \xrightarrow[\mathrm{bc}^!_{g_2}]{} & \overline{g}_2^! f g_{4!} & \xleftarrow[\mathrm{bc}_{!g_4}]{} & \overline{g}_2^! \overline{g}_{4!} f \end{array} \tag{3.14b}$$

Proof 1. If F is of type (C2) then Axiom (M7) holds by definition. If F is of type (C1) we may replace c and $\overline{c}$ by their compactifications since both statements are local. Thus we may assume that the g_i and $\overline{g_i}$, $i = 1, 2$, are all proper. By Axiom (M3), both statements are true for bc^*, hence also for bc_*. But then Axiom (M6) implies that they are true for $\mathrm{bc}_!$ hence also for $\mathrm{bc}^!$.

2. This follows immediately from part 1.

3. The proof of the commutativity is similar for the two diagrams and we will give it only for the second one.

   By adjointness, (3.14b) may be expanded as follows ($g := g_4 g_1 = g_2 g_3$):



The subdiagrams ① and ③ commute by part 1, ② commutes by the definition of (1.11), and ④ commutes by the definition of $\mathrm{bc}_!$. The rest of the diagram commutes by the naturality of the corresponding morphisms. □

Lemma 3.11 *F satisfies Axioms (M8) and (M9).*

Proof If F is of type (C2) then these axioms are satisfied by definition. If F is of type (C1) then we may replace c and $\overline{c}$ by their compactifications since the statements in both axioms are local. Thus we may assume the g, $\overline{g}$, g_i and $\overline{g}_i$, $i = 1, \dots, 4$, to be proper. Now, Axiom (M6) implies that $\mathrm{bc}_! = \mathrm{bc}_*^{-1}$ hence turning the diagram (3.5) clockwise by ninety degrees yields exactly (3.14a) which is commutative as we have shown.

For (3.6), we replace $\mathrm{bc}_!$ by bc_*^{-1} and use the decomposition of proj in 1.§3 to get the following diagram (via the adjointness relation $\overline{g}^* \dashv \overline{g}_* = \overline{g}_!$):

$$
\begin{array}{ccccccc}
f g^! \mathscr{A} \otimes \overline{g}^* f \mathscr{B} & \xrightarrow{\mathrm{bc}_g^*} & f g^! \mathscr{A} \otimes f g^* \mathscr{B} & \xrightarrow{t} & f(g^! \mathscr{A} \otimes g^* \mathscr{B}) & & \\
\uparrow{\scriptstyle \mathrm{adj}} & & \uparrow{\scriptstyle \mathrm{adj}} & & \uparrow{\scriptstyle \mathrm{adj}} & & \\
f g^* g_* g^! \mathscr{A} \otimes \overline{g}^* f \mathscr{B} & \xrightarrow{\mathrm{bc}_g^*} & f g^* g_* g^! \mathscr{A} \otimes f g^* \mathscr{B} & \xrightarrow{t} & f(g^* g_* g^! \mathscr{A} \otimes g^* \mathscr{B}) & & \\
\uparrow{\scriptstyle \mathrm{bc}_g^*} & \nearrow{\scriptstyle \mathrm{bc}_g^* \otimes \mathrm{bc}_g^*} & & & & \searrow{\scriptstyle \cong} & \\
\overline{g}^* f g_* g^! \mathscr{A} \otimes \overline{g}^* f \mathscr{B} & \xleftarrow{\cong} & \overline{g}^*(f g_* g^! \mathscr{A} \otimes f \mathscr{B}) & \xrightarrow{t} & \overline{g}^* f(g_* g^! \mathscr{A} \otimes \mathscr{B}) & \xrightarrow{\mathrm{bc}_g^*} & f g^*(g_* g^! \mathscr{A} \otimes \mathscr{B})
\end{array}
$$

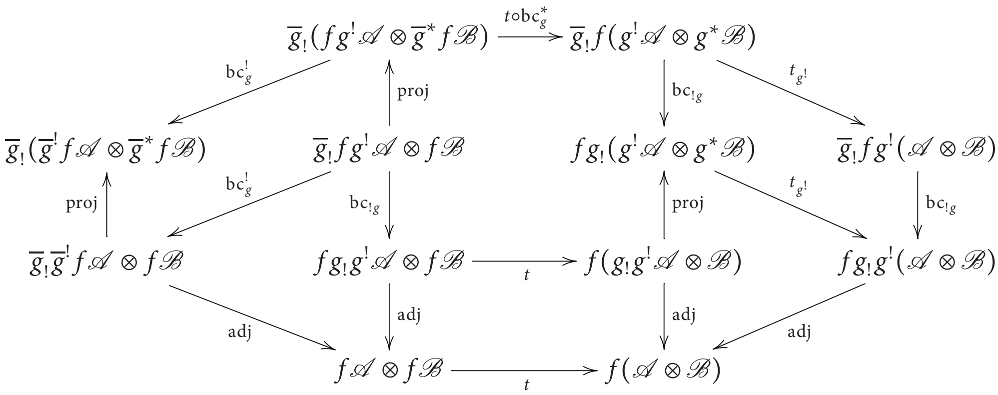
The upper half commutes by the naturality of the horizontal morphisms, the trapezoid in the lower half commutes by Axiom (M4), while the triangle on the left clearly commutes. Hence the whole diagram is commutative as claimed. □

Lemma 3.12 *For each $g : Z_1 \to Z_2$ in $Mor(c)$ and $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(Z_2)$, the following diagrams are commutative:*

$$
\begin{array}{ccccc}
f g^! \mathscr{A} \otimes \overline{g}^* f \mathscr{B} & \xrightarrow{\mathrm{bc}_g^*} & f g^! \mathscr{A} \otimes f g^* \mathscr{B} & \xrightarrow{t_{g^!} \circ t} & f g^!(\mathscr{A} \otimes \mathscr{B}) \\
\downarrow{\scriptstyle \mathrm{bc}_g^!} & & & & \downarrow{\scriptstyle \mathrm{bc}_g^!} \\
\overline{g}^! f \mathscr{A} \otimes \overline{g}^* f \mathscr{B} & \xrightarrow{t_{\overline{g}^!}} & \overline{g}^!(f \mathscr{A} \otimes f \mathscr{B}) & \xrightarrow{t} & \overline{g}^! f(\mathscr{A} \otimes \mathscr{B})
\end{array}
\qquad (3.15a)
$$

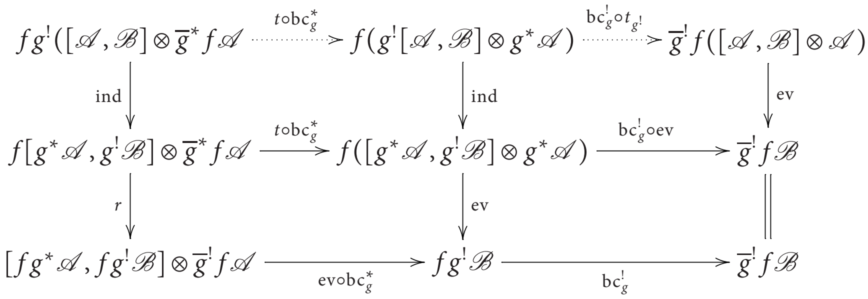
$$
\begin{array}{ccccc}
f g^! \,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) & \xrightarrow{\mathrm{bc}_g^!} & \overline{g}^! f \,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) & \xrightarrow{r_{Z_2}} & \overline{g}^! \,\mathbf{R}\underline{\mathrm{Hom}}(f \mathscr{A}, f \mathscr{B}) \\
\downarrow{\scriptstyle \mathrm{ind}\ \cong} & & & & \downarrow{\scriptstyle \cong\ \mathrm{ind}} \\
f \,\mathbf{R}\underline{\mathrm{Hom}}(g^* \mathscr{A}, g^! \mathscr{B}) & \xrightarrow{r_{Z_1}} & \mathbf{R}\underline{\mathrm{Hom}}(f g^* \mathscr{A}, f g^! \mathscr{B}) & \xrightarrow{\mathrm{bc}_g^* \times \mathrm{bc}_g^!} & \mathbf{R}\underline{\mathrm{Hom}}(\overline{g}^* f \mathscr{A}, \overline{g}^! f \mathscr{B})
\end{array}
\qquad (3.15b)
$$

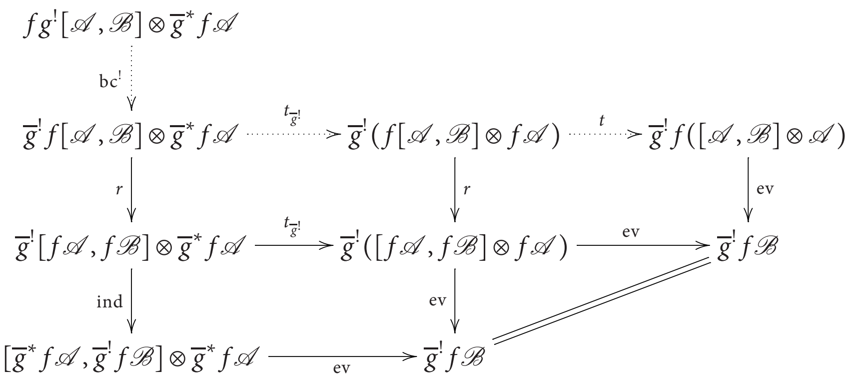
Proof By adjointness, (3.15a) may be expanded as follows:



The square in the upper half commutes by Axiom (M9), while the rest of the diagram is clearly commutative.

For diagram (3.15b), notice that the two following diagrams commute by the definition of the morphisms involved (we abbreviate $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$ by $[-,-]$):





Now, following the left vertical column and the bottom row in the two diagrams gives, by adjointness, the two maps

$$fg^! \,\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A},\mathscr{B}) \longrightarrow \mathbf{R}\underline{\mathrm{Hom}}(\overline{g}^* f\mathscr{A}, \overline{g}^! f\mathscr{B})$$

of (3.15b), respectively. On the other hand, we know that the dotted paths are equal by the commutativity of (3.15a). Therefore (3.15b) must commute also. □

We need more notation.

Notation 3.13 1. For $Z \in Ob(c)$ and $\mathscr{A} \in \mathfrak{D}^b_{ctf}(Z)$, we denote by $h^0_Z : \mathrm{H}^0(Z, \mathscr{A}) \to \mathrm{H}^0(\overline{Z}, f_Z\mathscr{A})$ the composition

$$\mathrm{Hom}(\Lambda_Z, \mathscr{A}) \xrightarrow{f_Z} \mathrm{Hom}(f_Z\Lambda_Z, f_Z\mathscr{A}) \xrightarrow{\iota_Z} \mathrm{Hom}(\Lambda_{\overline{Z}}, f_Z\mathscr{A}).$$

2. Given $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(X)$, we denote by $\prod_{\mathscr{A},\mathscr{B}} : f_X\mathscr{A} \boxtimes f_X\mathscr{B} \to f_{X\times X}(\mathscr{A} \boxtimes \mathscr{B})$ the composition

$$\overline{p}_1^* f_X\mathscr{A} \otimes \overline{p}_2^* f_X\mathscr{B} \xrightarrow{\mathrm{bc}^*_{p_1} \otimes \mathrm{bc}^*_{p_2}} f_{X\times X}p_1^*\mathscr{A} \otimes f_{X\times X}p_2^*\mathscr{B} \xrightarrow{t_{X\times X}} f_{X\times X}(p_1^*\mathscr{A} \otimes p_2^*\mathscr{B}).$$

Lemma 3.14 *1. For $g : Z_1 \to Z_2$ in $Mor(c)$, $\iota_{Z_1} : \Lambda_{\overline{Z}_1} \to f_{Z_1}\Lambda_{Z_1}$ decomposes as follows:*

$$\overline{g}^*\Lambda_{\overline{Z}_2} \xrightarrow{\iota_{Z_2}} \overline{g}^* f_{Z_2}\Lambda_{Z_2} \xrightarrow{\mathrm{bc}^*_g} f_{Z_1}g^*\Lambda_{Z_2}.$$

2. *For $Z \in Ob(c)$ and $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(Z)$, the composition*

$$\mathrm{H}^0(Z, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B})) \xrightarrow{h^0_Z} \mathrm{H}^0(\overline{Z}, f_Z\, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B})) \xrightarrow{r_Z} \mathrm{H}^0(\overline{Z}, \mathbf{R}\underline{\mathrm{Hom}}(f_Z\mathscr{A}, f_Z\mathscr{B}))$$

may be identified with $f_Z : \mathrm{Hom}(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}(f_Z\mathscr{A}, f_Z\mathscr{B})$.

Proof 1. This follows immediately from Axiom (M3).

2. Let $\zeta \in \mathrm{Hom}(\mathscr{A}, \mathscr{B})$ and denote by $\zeta' \in \mathrm{Hom}(\Lambda, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}))$ the morphism corresponding to ζ. Then $r_Z \circ h^0_Z(\zeta') \in \mathrm{Hom}(\Lambda_{\overline{Z}}, \mathbf{R}\underline{\mathrm{Hom}}(f_Z\mathscr{A}, f_Z\mathscr{B}))$ corresponds by adjointness to the dotted path in the following diagram:

$$\begin{array}{ccccc}
\Lambda_{\overline{Z}} \otimes f_Z\mathscr{A} & \overset{\iota_Z}{\dashrightarrow} & f_Z\Lambda_Z \otimes f_Z\mathscr{A} & \overset{f_Z\zeta' \otimes \mathbb{1}}{\dashrightarrow} & f_Z\, \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) \otimes f_Z\mathscr{A} \\
 & & \downarrow t_Z & & \vdots\, t_Z \\
 & & f_Z(\Lambda_Z \otimes \mathscr{A}) & \xrightarrow{f_Z(\zeta' \otimes \mathbb{1})} & f_Z(\mathbf{R}\underline{\mathrm{Hom}}(\mathscr{A}, \mathscr{B}) \otimes \mathscr{A}) \\
 & & \downarrow \cong & & \vdots\, \mathrm{ev} \\
 & & f_Z\mathscr{A} & \xrightarrow[f_Z\zeta]{} & f_Z\mathscr{B}
\end{array}$$

The upper square commutes by the naturality of t_Z while the lower square commutes by the definition of ζ' therefore the whole diagram is commutative. With Axiom (M2) we conclude that $f_Z\zeta$ and $r_Z \circ h^0_Z(\zeta')$ correspond by adjointness to the same morphism hence they are equal themselves as claimed. □

Lemma 3.15 *1. For $g : Z_1 \to Z_2$ in $Mor(c)$ and $\mathscr{A} \in \mathfrak{D}^b_{ctf}(Z_1)$, the composition*

$$\mathrm{H}^0(Z_2, g_*\mathscr{A}) \xrightarrow{h^0_{Z_2}} \mathrm{H}^0(\overline{Z_2}, f_{Z_2}g_*\mathscr{A}) \xrightarrow{\mathrm{bc}^*_g} \mathrm{H}^0(\overline{Z_2}, \overline{g}_*f_{Z_1}\mathscr{A})$$

may be identified with $h^0_{Z_1} : \mathrm{H}^0(Z_1, \mathscr{A}) \to \mathrm{H}^0(\overline{Z_1}, f_{Z_1}\mathscr{A})$.

2. *For $\mathscr{A}, \mathscr{B} \in \mathfrak{D}^b_{ctf}(X)$, the composition*

$$\overline{\Delta}^*(f_X\mathscr{A} \boxtimes f_X\mathscr{B}) \xrightarrow{\Pi_{\mathscr{A},\mathscr{B}}} \overline{\Delta}^* f_{X\times X}(\mathscr{A}\boxtimes\mathscr{B}) \xrightarrow{\mathrm{bc}^*_\Delta} f_X\Delta^*(\mathscr{A}\boxtimes\mathscr{B})$$

may be identified with $t_X : f_X\mathscr{A} \otimes f_X\mathscr{B} \to f_X(\mathscr{A}\otimes\mathscr{B})$.

Proof 1. Let $\zeta \in \mathrm{Hom}(\Lambda_{Z_2}, g_*\mathscr{A})$ and denote by $\zeta' \in \mathrm{Hom}(\Lambda_{Z_1}, \mathscr{A})$ the morphism corresponding to ζ. Then $\mathrm{bc}^*_g \circ h^\circ_{Z_2}(\zeta) \in \mathrm{Hom}(\Lambda_{\overline{Z_2}}, \overline{g}_* f_{Z_1}\mathscr{A})$ corresponds by adjointness to the dotted path in the following diagram:

$$\begin{array}{ccccc}
\overline{g}^*\Lambda_{\overline{Z_2}} & \xrightarrow{\iota_{Z_2}} & \overline{g}^* f_{Z_2}\Lambda_{Z_2} & \xrightarrow{\overline{g}^* f_{Z_2}\zeta} & \overline{g}^* f_{Z_2} g_*\mathscr{A} \\
 & & \downarrow{\scriptstyle \mathrm{bc}^*_g} & & \downarrow{\scriptstyle \mathrm{bc}^*_g} \\
\downarrow{\scriptstyle\cong} & & f_{Z_1} g^*\Lambda_{Z_2} & \xrightarrow{f_{Z_1} g^*\zeta} & f_{Z_1} g^* g_*\mathscr{A} \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle \mathrm{adj}} \\
\Lambda_{\overline{Z_1}} & \xrightarrow{\iota_{Z_1}} & f_{Z_1}\Lambda_{Z_1} & \xrightarrow{f_{Z_1}\zeta'} & f_{Z_1}\mathscr{A}
\end{array}$$

The left half of the diagram is commutative by 3.14.1, the upper right square by the naturality of bc^*_g, and the lower right square by the definition of ζ'. Hence the whole diagram commutes which is exactly what the lemma asserts.

2. Consider the following diagram:

$$\begin{array}{ccccc}
f\mathscr{A}\otimes f\mathscr{B} & = & = & = & f\mathscr{A}\otimes f\mathscr{B} \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
\overline{\Delta}^*\overline{p}_1^* f\mathscr{A}\otimes\overline{\Delta}^*\overline{p}_2^* f\mathscr{B} & \xrightarrow{\mathrm{bc}^*_{p_1}\otimes\mathrm{bc}^*_{p_2}} & \overline{\Delta}^* f p_1^*\mathscr{A}\otimes\overline{\Delta}^* f p_2^*\mathscr{B} & \xrightarrow{\mathrm{bc}^*_\Delta\otimes\mathrm{bc}^*_\Delta} & f\Delta^* p_1^*\mathscr{A}\otimes f\Delta^* p_2^*\mathscr{B} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle t} \\
\overline{\Delta}^*(\overline{p}_1^* f\mathscr{A}\otimes\overline{p}_2^* f\mathscr{B}) & \xrightarrow{\mathrm{bc}^*_{p_1}\otimes\mathrm{bc}^*_{p_2}} & \overline{\Delta}^*(f p_1^*\mathscr{A}\otimes f p_2^*\mathscr{B}) & & f(\Delta^* p_1^*\mathscr{A}\otimes\Delta^* p_2^*\mathscr{B}) \\
\| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle\cong} \\
\overline{\Delta}^*(f\mathscr{A}\boxtimes f\mathscr{B}) & \xrightarrow{\Pi_{\mathscr{A},\mathscr{B}}} & \overline{\Delta}^* f(\mathscr{A}\boxtimes\mathscr{B}) & \xrightarrow{\mathrm{bc}^*_\Delta} & f\Delta^*(\mathscr{A}\boxtimes\mathscr{B})
\end{array}$$

The top square of this diagram commutes by Axiom (M3), and the lower right square by Axiom (M4). The bottom left square is commutative by the definition of $\prod_{\mathscr{A},\mathscr{B}}$, while the square above it commutes by the naturality of the vertical isomorphisms (1.1). Thus the whole diagram is commutative. □

Lemma 3.16 *1. Denote by α and β, respectively, the following two base change morphisms:*

$$\alpha : \overline{p}_1^* K_{\overline{X}} = \overline{p}_1^*\pi^!_{\overline{X}}\Lambda_k \xrightarrow{\mathrm{bc}\ (1.15)} \overline{p}_2^!\pi^*_{\overline{X}}\Lambda_k \xrightarrow{\cong} \overline{p}_2^!\Lambda_{\overline{X}},$$

$$\beta : p_1^* K_X = p_1^*\pi^!_X\Lambda_k \xrightarrow{\mathrm{bc}\ (1.15)} p_2^!\pi^*_X\Lambda_k \xrightarrow{\cong} p_2^!\Lambda_X.$$

With this notation, the following diagram commutes:

$$
\begin{array}{ccccc}
\overline{p}_1^* f_X K_X & \xrightarrow{\tilde{\pi}_X} & \overline{p}_1^* K_{\overline{X}} & \xrightarrow{\alpha} & \overline{p}_2^! \Lambda_{\overline{X}} \\
\downarrow{\scriptstyle \mathrm{bc}^*_{p_1}} & & & & \downarrow{\scriptstyle \iota_X} \\
f_{X\times X} p_1^* K_X & \xrightarrow[\beta]{} & f_{X\times X} p_2^! \Lambda_X & \xrightarrow[\mathrm{bc}^!_{p_2}]{} & \overline{p}_2^! f_X \Lambda_X
\end{array}
\tag{3.16}
$$

2. *For $\mathscr{A} \in \mathfrak{D}^b_{ctf}(X)$, the following diagram commutes:*

$$
\begin{array}{ccc}
f_{X\times X} p_1^* K_X \otimes f_{X\times X} p_2^* \mathscr{A} & \xrightarrow{(1.17)\circ t_{X\times X}} & f_{X\times X} p_2^! \mathscr{A} \\
\uparrow{\scriptstyle \mathrm{bc}^*_{p_1} \otimes \mathrm{bc}^*_{p_2}} & & \downarrow{\scriptstyle \mathrm{bc}^!_{p_2}} \\
\overline{p}_1^* f_X K_X \otimes \overline{p}_2^* f_X \mathscr{A} & \xrightarrow[(1.17)\circ\tilde{\pi}_X]{} & \overline{p}_2^! f_X \mathscr{A}
\end{array}
$$

Proof 1. Consider first the following diagram:

$$
\begin{array}{ccccc}
\overline{p}_{2!}\overline{p}_1^* f \pi_X^! \Lambda_k & \xrightarrow{\tilde{\pi}_X} & \overline{p}_{2!}\overline{p}_1^* \pi_{\overline{X}}^! \Lambda_k & \xrightarrow[\cong]{\mathrm{bc}\ (1.15)} & \overline{p}_{2!}\overline{p}_2^! \pi_{\overline{X}}^! \Lambda_k \\
\uparrow{\scriptstyle \mathrm{bc}\ (1.13)\ \cong} & & \uparrow{\scriptstyle \mathrm{bc}\ (1.13)\ \cong} & & \downarrow{\scriptstyle \mathrm{adj}} \\
\pi_{\overline{X}}^* \pi_{\overline{X}!} f \pi_X^! \Lambda_k & \xrightarrow{\tilde{\pi}_X} & \pi_{\overline{X}}^* \pi_{\overline{X}!} \pi_{\overline{X}}^! \Lambda_k & \xrightarrow{\mathrm{adj}} & \pi_{\overline{X}}^* \Lambda_k \\
\downarrow{\scriptstyle \mathrm{bc}_{!\pi_X}} & & & & \downarrow{\scriptstyle \iota^{-1}} \\
\pi_{\overline{X}}^* f \pi_{X!} \pi_X^! \Lambda_k & & \xrightarrow[\mathrm{adj}]{\qquad\qquad\qquad} & & \pi_{\overline{X}}^* f \Lambda_k
\end{array}
$$

The upper left square commutes by the naturality of (1.13), the right upper square by the definition of (1.15), while the commutativity of the lower square follows easily from 3.10.2. Hence the whole diagram is commutative which allows us to replace the (morphism adjoint to the) composition $\iota_X \circ \alpha \circ \tilde{\pi}_X$ in (3.16) by the dotted path in the following diagram:

$$
\begin{array}{ccccc}
\overline{p}_{2!}\overline{p}_1^* f \pi_X^! \Lambda_k & \xrightarrow[\cong]{\mathrm{bc}\ (1.13)^{-1}} & \pi_{\overline{X}}^* \pi_{\overline{X}!} f \pi_X^! \Lambda_k & & \\
 & & \downarrow{\scriptstyle \mathrm{bc}_{!\pi_X}} & & \\
\downarrow{\scriptstyle \mathrm{bc}_{!p_2} \circ \mathrm{bc}^*_{p_1}} & & \pi_{\overline{X}}^* f \pi_{X!} \pi_X^! \Lambda_k & \xrightarrow{\mathrm{adj}} & \pi_{\overline{X}}^* f \Lambda_k \\
 & & \downarrow{\scriptstyle \mathrm{bc}^*_{\pi_X}} & & \downarrow{\scriptstyle \mathrm{bc}^*_{\pi_X}} \\
f p_{2!} p_1^* \pi_X^! \Lambda_k & \xrightarrow[\mathrm{bc}\ (1.13)^{-1}]{} & f \pi_X^* \pi_{X!} \pi_X^! \Lambda_k & \xrightarrow[\mathrm{adj}]{} & f \pi_X^* \Lambda_k
\end{array}
$$

The right square commutes by the naturality of $\mathrm{bc}^*_{\pi_X}$ while the left square commutes by Axiom (M8) (cf. 3.11). A similar argument shows that the solid path $\overline{p}_{2!}\overline{p}_1^* f \pi_X^! \Lambda_k \to f \pi_X^* \Lambda_k$ in this diagram corresponds by adjointness to the composition $\mathrm{bc}^!_{p_2} \circ \beta \circ \mathrm{bc}^*_{p_1}$ in (3.16), which completes the proof.

2. The diagram extends to the following one:

$$
\begin{array}{ccccccc}
\overline{p}_1^* f K_X \otimes \overline{p}_2^* f\mathscr{A} & \xrightarrow{\alpha\circ\tilde{\pi}_X} & \overline{p}_2^! \Lambda_{\overline{X}} \otimes \overline{p}_2^* f\mathscr{A} & \xrightarrow{t_{\overline{p}_2^!}} & \overline{p}_2^!(\Lambda_k \otimes f\mathscr{A}) & \xrightarrow{\cong} & \overline{p}_2^! f\mathscr{A} \\
 & & \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota_X} & & \uparrow{\scriptstyle \cong} \\
\downarrow{\scriptstyle \mathrm{bc}_{p_1}^*} & & \overline{p}_2^! f\Lambda_X \otimes \overline{p}_2^* f\mathscr{A} & \xrightarrow{t_{\overline{p}_2^!}} & \overline{p}_2^!(f\Lambda_X \otimes f\mathscr{A}) & \xrightarrow{t_X} & \overline{p}_2^! f(\Lambda_X \otimes \mathscr{A}) \\
 & & \uparrow{\scriptstyle \mathrm{bc}_{p_2}^!} & & & \nearrow{\scriptstyle \mathrm{bc}_{p_2}^!} & \uparrow{\scriptstyle \mathrm{bc}_{p_2}^!} \\
f p_1^* K_X \otimes \overline{p}_2^* f\mathscr{A} & \xrightarrow{\beta} & f p_2^! \Lambda_X \otimes \overline{p}_2^* f\mathscr{A} & & & & f p_2^!(\Lambda_X \otimes \mathscr{A}) \\
\downarrow{\scriptstyle \mathrm{bc}_{p_2}^*} & & \downarrow{\scriptstyle \mathrm{bc}_{p_2}^*} & & & & \uparrow{\scriptstyle \cong} \\
f p_1^* K_X \otimes f p_2^* \mathscr{A} & \xrightarrow[\beta]{} & f p_2^! \Lambda_X \otimes f p_2^* \mathscr{A} & \xrightarrow[t_{p_2^!}\circ t_{X\times X}]{} & f p_2^!(\Lambda_X \otimes \mathscr{A}) & \xrightarrow[\cong]{} & f p_2^! \mathscr{A}
\end{array}
$$

The upper left square commutes by part 1, while the lower left and the upper middle square commute by the naturality of the morphisms involved. Next, the top right square commutes by Axiom (M2), the triangle in the right bottom corner is clearly commutative, while the remaining trapezoid commutes by 3.12. Thus the whole diagram is commutative. □

§4 Naturality of the trace map

With all these compatibilities proved in the last paragraph we may now proceed to the general statement concerning the naturality of the trace morphism with respect to cohomological morphisms.

Proposition 3.17 *Let $F = (\{f_Z, t_Z\}, \{\mathrm{bc}_g^*, \mathrm{bc}_g^!\}, \iota)$ be a cohomological morphism from $c : C \to X \times X$ to $\overline{c}$ and let $\mathscr{F} \in \mathfrak{D}_{ctf}^b(X)$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
f_C\, \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) & \xrightarrow{f_c \underline{\mathrm{tr}}_c} & f_C \Delta'_* K_{\mathrm{Fix}(c)} \\
\downarrow{\scriptstyle (\mathrm{bc}_{c_1}^* \times \mathrm{bc}_{c_2}^!)\circ r_C} & & \downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}\circ \mathrm{bc}_{*\Delta'}} \\
\mathbf{R}\underline{\mathrm{Hom}}(\overline{c}_1^* f_X\mathscr{F}, \overline{c}_2^! f_X\mathscr{F}) & \xrightarrow[\underline{\mathrm{tr}}_{\overline{c}}]{} & \overline{\Delta}'_* K_{\mathrm{Fix}(\overline{c})}
\end{array}
\qquad (3.17)
$$

The proof of this proposition will be given in three steps.

Step 1 *We may assume that F is a good cohomological morphism.*

Proof We have to check that the vertical morphisms in (3.17) are compatible with composition. To this end, let $(\{f_{\overline{Z}}, t_{\overline{Z}}\}, \{\mathrm{bc}_{\overline{g}}^*, \mathrm{bc}_{\overline{g}}^!\}, \overline{\iota}) : \overline{c} \to \overline{\overline{c}}$ be another cohomological morphism. For the left vertical morphism the compatibility with respect to composition follows from the commutativity of the following diagram, i. e. from the naturality of $r_{\overline{C}}$ and from the fact that r is compatible with composition.

$$
\begin{array}{ccccc}
f_{\overline{C}} f_C[c_1^*\mathscr{F}, c_2^!\mathscr{F}] & \xrightarrow{r_C} & f_{\overline{C}}[f_C c_1^*\mathscr{F}, f_C c_2^!\mathscr{F}] & \xrightarrow{\mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2}} & f_{\overline{C}}[\overline{c}_1^* f_X\mathscr{F}, \overline{c}_2^! f_X\mathscr{F}] \\
\downarrow{\scriptstyle r_C} & \swarrow{\scriptstyle r_{\overline{C}}} & & \swarrow{\scriptstyle r_{\overline{C}}} & \downarrow{\scriptstyle (\mathrm{bc}^*_{\overline{c}_1}\times \mathrm{bc}^!_{\overline{c}_2})\circ r_{\overline{C}}} \\
[f_{\overline{C}} f_C c_1^*\mathscr{F}, f_{\overline{C}} f_C c_2^!\mathscr{F}] & \xrightarrow[\mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2}]{} & [f_{\overline{C}}\overline{c}_1^* f_X\mathscr{F}, f_{\overline{C}}\overline{c}_2^! f_X\mathscr{F}] & \xrightarrow[\mathrm{bc}^*_{\overline{c}_1}\times \mathrm{bc}^!_{\overline{c}_2}]{} & [\overline{\overline{c}}_1^* f_{\overline{X}} f_X\mathscr{F}, \overline{\overline{c}}_2^! f_{\overline{X}} f_X\mathscr{F}]
\end{array}
$$

(Here, as before, $[-,-]$ abbreviates $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$.) Similarly, for the right vertical morphism, the claim follows from the naturality of $\mathrm{bc}^*_{\overline{\Delta}'}$ and from the fact that $\tilde{\pi}$ is compatible with composition. □

Hence, from now on we will only consider good cohomological morphisms F and may thus apply the results of the previous paragraph.

Step 2 *The proposition holds in the case $c = \mathbb{1}_{X\times X}$ and $\overline{c} = \mathbb{1}_{\overline{X}\times\overline{X}}$.*

Proof This choice of c and $\overline{c}$ makes the global trace morphism somewhat easier to handle. Thus the diagram (3.17) decomposes as follows:

$$
\begin{array}{ccccc}
f_{X\times X}\,\mathbf{R}\underline{\mathrm{Hom}}(p_1^*\mathscr{F}, p_2^!\mathscr{F}) & \xleftarrow[\cong]{(1.18)} & f_{X\times X}(\mathbb{D}\mathscr{F}\boxtimes\mathscr{F}) & \xrightarrow{\mathrm{ev}\circ\mathrm{adj}} & f_{X\times X}\Delta_* K_X \\
\downarrow{\scriptstyle (\mathrm{bc}^*_{p_1}\times\mathrm{bc}^!_{p_2})\circ r_{X\times X}} & & \uparrow{\scriptstyle \Pi_{\mathbb{D}\mathscr{F},\mathscr{F}}} & & \downarrow{\scriptstyle \tilde{\pi}_X\circ\mathrm{bc}_{*\Delta}} \\
\mathbf{R}\underline{\mathrm{Hom}}(\overline{p}_1^* f_X\mathscr{F}, \overline{p}_2^! f_X\mathscr{F}) & \xleftarrow[(1.18)\circ d_X]{\cong} & f_X\mathbb{D}\mathscr{F}\boxtimes f_X\mathscr{F} & \xrightarrow[\mathrm{ev}\circ d_X\circ\mathrm{adj}]{} & \overline{\Delta}_* K_{\overline{X}}
\end{array}
\qquad (3.18)
$$

Notice that we have inserted d_X twice in the bottom row which we are allowed to do since it is an isomorphism by Axiom (M5). We will now in turn prove the commutativity of the two inner squares.

We expand the left (adjoint) square as follows:

$$
\begin{array}{ccccc}
f p_2^!\mathscr{F} & \xleftarrow{(1.17)} & f(p_1^*K_X\otimes p_2^*\mathscr{F}) & \xleftarrow{\mathrm{ev}\circ t_{p_1^*}\circ t} & f p_1^*\mathscr{F}\otimes f(p_1^*\mathbb{D}\mathscr{F}\otimes p_2^*\mathscr{F}) \\
\| & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle t} \\
f p_2^!\mathscr{F} & \xleftarrow{(1.17)\circ t} & f p_1^*K_X\otimes f p_2^*\mathscr{F} & \xleftarrow{\mathrm{ev}\circ t_{p_1^*}\circ t} & f p_1^*\mathscr{F}\otimes f p_1^*\mathbb{D}\mathscr{F}\otimes f p_2^*\mathscr{F} \\
\downarrow{\scriptstyle \mathrm{bc}^!_{p_2}} & & \uparrow{\scriptstyle \mathrm{bc}^*_{p_1}\otimes\mathrm{bc}^*_{p_2}} & & \uparrow{\scriptstyle \mathrm{bc}^*_{p_1}\otimes\mathrm{bc}^*_{p_1}\otimes\mathrm{bc}^*_{p_2}} \\
\overline{p}_2^! f\mathscr{F} & \xleftarrow{(1.17)\circ\tilde{\pi}_X} & \overline{p}_1^* f K_X\otimes\overline{p}_2^* f\mathscr{F} & \xleftarrow{\mathrm{ev}\circ t\circ t_{\overline{p}_1^*}} & \overline{p}_1^* f\mathscr{F}\otimes\overline{p}_1^* f\mathbb{D}\mathscr{F}\otimes\overline{p}_2^* f\mathscr{F}
\end{array}
$$

The upper left square is clearly commutative. The commutativity of the upper (resp. lower) right square follows immediately from Axiom (M1) (resp. (M4)), while the lower left square commutes by 3.16.2.

Using the definition of bc_* the right (adjoint) square of (3.18) may be expanded as follows:

$$
\begin{array}{ccccc}
\overline{\Delta}^* f(\mathbb{D}\mathscr{F}\boxtimes\mathscr{F}) & \xrightarrow{\mathrm{bc}^*_{\Delta}} & f\Delta^*(\mathbb{D}\mathscr{F}\boxtimes\mathscr{F}) & \xrightarrow{\mathrm{ev}} & f K_X \\
\uparrow{\scriptstyle \overline{\Delta}^*\Pi_{\mathbb{D}\mathscr{F},\mathscr{F}}} & & \| & & \| \\
f\mathbb{D}\mathscr{F}\otimes f\mathscr{F} & \xrightarrow[t]{} & f(\mathbb{D}\mathscr{F}\otimes\mathscr{F}) & \xrightarrow[\mathrm{ev}]{} & f K_X \\
\| & & & & \downarrow{\scriptstyle \tilde{\pi}_X} \\
f\mathbb{D}\mathscr{F}\otimes f\mathscr{F} & \xrightarrow[d_X]{} & \mathbb{D} f\mathscr{F}\otimes f\mathscr{F} & \xrightarrow[\mathrm{ev}]{} & K_{\overline{X}}
\end{array}
$$

The upper right square is clearly commutative. The upper left square is commutative by 3.15.2, while the commutativity of the lower half follows from the definition of d_X. □

Step 3 *Proposition 3.17 is true (the general case).*

Proof The following decomposition of (3.17) allows us to apply the previous step ($[-,-]$ abbreviates $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$):

$$
\begin{array}{ccccccc}
f[c_1^*\mathscr{F}, c_2^!\mathscr{F}] & \xleftarrow[\cong]{\text{ind}} & fc^![p_1^*\mathscr{F}, p_2^!\mathscr{F}] & \xrightarrow{fc^!\underline{\mathrm{tr}}_{1_{X\times X}}} & fc^!\Delta_* K_X & \xleftarrow[\cong]{\text{bc (1.14)}} & f\Delta'_* K_{\mathrm{Fix}(c)} \\
\Big\downarrow{\scriptstyle (\mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2})\circ r_C} & & \Big\downarrow{\scriptstyle \alpha\circ \mathrm{bc}^!_c} & & \Big\downarrow{\scriptstyle \beta\circ\mathrm{bc}^!_c} & & \Big\downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}\circ \mathrm{bc}_{*\Delta'}} \\
[\overline{c}_1^* f\mathscr{F}, \overline{c}_2^! f\mathscr{F}] & \xleftarrow[\text{ind}]{\cong} & \overline{c}^![\overline{p}_1^* f\mathscr{F}, \overline{p}_2^! f\mathscr{F}] & \xrightarrow[\overline{c}^!\underline{\mathrm{tr}}_{1_{\overline{X}\times\overline{X}}}]{} & \overline{c}^!\overline{\Delta}_* K_{\overline{X}} & \xleftarrow[\text{bc (1.14)}]{\cong} & \overline{\Delta}'_* K_{\mathrm{Fix}(\overline{c})}
\end{array}
\tag{3.19}
$$

Here, α and β are the left and right vertical arrows of (3.18), respectively. Hence the commutativity of the middle square follows from that of (3.18). It thus remains to prove the commutativity of the left and right squares of (3.19).

The left square may be expanded as follows:

$$
\begin{array}{ccc}
f\,\mathbf{R}\underline{\mathrm{Hom}}(c^*p_1^*\mathscr{F}, c^!p_2^!\mathscr{F}) & \xleftarrow[\text{ind}]{\cong} & fc^!\,\mathbf{R}\underline{\mathrm{Hom}}(p_1^*\mathscr{F}, p_2^!\mathscr{F}) \\
\Big\downarrow{\scriptstyle (\mathrm{bc}^*_c\times\mathrm{bc}^!_c)\circ r_C} & & \Big\downarrow{\scriptstyle r_{X\times X}\circ\mathrm{bc}^!_c} \\
\mathbf{R}\underline{\mathrm{Hom}}(\overline{c}^* f p_1^*\mathscr{F}, \overline{c}^! f p_2^!\mathscr{F}) & \xleftarrow[\text{ind}]{\cong} & \overline{c}^!\,\mathbf{R}\underline{\mathrm{Hom}}(f p_1^*\mathscr{F}, f p_2^!\mathscr{F}) \\
\Big\downarrow{\scriptstyle \mathrm{bc}^*_{p_1}\times\mathrm{bc}^!_{p_2}} & & \Big\downarrow{\scriptstyle \mathrm{bc}^*_{p_1}\times\mathrm{bc}^!_{p_2}} \\
\mathbf{R}\underline{\mathrm{Hom}}(\overline{c}^*\overline{p}_1^* f\mathscr{F}, \overline{c}^!\overline{p}_2^! f\mathscr{F}) & \xleftarrow[\text{ind}]{\cong} & \overline{c}^!\,\mathbf{R}\underline{\mathrm{Hom}}(\overline{p}_1^* f\mathscr{F}, \overline{p}_2^! f\mathscr{F})
\end{array}
\tag{3.20}
$$

Here, we have used Axioms (M3) and (M7) (cf. 3.10.1), i. e. the fact that $\mathrm{bc}^*_{c_1}$ factors as $\mathrm{bc}^*_c \circ \mathrm{bc}^*_{p_1}$ and $\mathrm{bc}^!_{c_2}$ factors as $\mathrm{bc}^!_{p_2}\circ\mathrm{bc}^!_c$. The upper square of (3.20) commutes by 3.12, while the lower square commutes by the naturality of ind. Hence the left square of (3.19) is commutative.

The right square of (3.19) may be expanded as follows:

$$
\begin{array}{ccccc}
fc^!\Delta_* K_X & \xleftarrow{\text{bc (1.14)}} & f\Delta'_* c'^! K_X & = & f\Delta'_* K_{\mathrm{Fix}(c)} \\
\Big\downarrow{\scriptstyle \mathrm{bc}_{*\Delta}\circ\mathrm{bc}^!_c} & & \Big\downarrow{\scriptstyle \mathrm{bc}^!_{c'}\circ\mathrm{bc}_{*\Delta'}} & & \Big\downarrow{\scriptstyle \mathrm{bc}_{*\Delta'}} \\
\overline{c}^!\overline{\Delta}_* f K_X & \xleftarrow[\text{bc (1.14)}]{} & \overline{\Delta}'_*\overline{c}'^! f K_X & \xleftarrow[\mathrm{bc}^!_{c'}]{} & \overline{\Delta}'_* f K_{\mathrm{Fix}(c)} \\
\Big\downarrow{\scriptstyle \tilde{\pi}_X} & & \Big\downarrow{\scriptstyle \tilde{\pi}_X} & & \Big\downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}} \\
\overline{c}^!\overline{\Delta}_* K_{\overline{X}} & \xleftarrow[\text{bc (1.14)}]{} & \overline{\Delta}'_*\overline{c}'^! K_{\overline{X}} & = & \overline{\Delta}'_* K_{\mathrm{Fix}(\overline{c})}
\end{array}
$$

The lower left square commutes by the naturality of (1.14), while the lower right square does so by 3.10.2. In the upper half the commutativity of the left square follows easily from Axiom (M8) (cf. 3.11), while the right square obviously commutes. Thus (3.19) and consequently (3.17) are commutative. □

We shall now interpret this general result in the context in which it is most useful to us. Fix a cohomological morphism $F : c \to \overline{c}$ as in the proposition, and fix also $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$. We associate with these data two maps

$$\begin{aligned} L_F = L_{F,\mathscr{F}} &: \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{Hom}_{\overline{c}}(f_X\mathscr{F}, f_X\mathscr{F}),\\ R_F = R_{F,\mathscr{F}} &: \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \longrightarrow \mathrm{H}^0(\mathrm{Fix}(\overline{c}), K_{\mathrm{Fix}(\overline{c})}),\end{aligned}$$

defined by

$$\begin{aligned} &L_F : \mathrm{Hom}(c_{2!}c_1^*\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{bc}^*_{c_1}\circ \mathrm{bc}_{!c_2}\circ f_X} \mathrm{Hom}(\overline{c}_{2!}\overline{c}_1^* f_X\mathscr{F}, f_X\mathscr{F}),\\ &R_F = \tilde{\pi}_{\mathrm{Fix}(c)} \circ h^0_{\mathrm{Fix}(c)}.\end{aligned}$$

Corollary 3.18 *Let $F = (\{f_Z, t_Z\}, \{\mathrm{bc}^*_g, \mathrm{bc}^!_g\}, \iota)$ be a cohomological morphism from $c : C \to X \times X$ to $\overline{c}$ and let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Hom}_c(\mathscr{F},\mathscr{F}) & \xrightarrow{\mathrm{tr}_c} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle L_F} & & \downarrow{\scriptstyle R_F} \\ \mathrm{Hom}_{\overline{c}}(f_X\mathscr{F}, f_X\mathscr{F}) & \xrightarrow[\mathrm{tr}_{\overline{c}}]{} & \mathrm{H}^0(\mathrm{Fix}(\overline{c}), K_{\mathrm{Fix}(\overline{c})}) \end{array}$$

Proof Consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^0(C, \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F})) & \xrightarrow{\mathrm{tr}_c} & \mathrm{H}^0(C, \Delta'_* K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle h^0_C} & & \downarrow{\scriptstyle h^0_C} \\ \mathrm{H}^0(\overline{C}, f_C\, \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F})) & \xrightarrow{f_C \mathrm{tr}_c} & \mathrm{H}^0(\overline{C}, f_C \Delta'_* K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle (\mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2})\circ r_C} & & \downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}\circ \mathrm{bc}_{*\Delta'}} \\ \mathrm{H}^0(\overline{C}, \mathbf{R}\underline{\mathrm{Hom}}(\overline{c}_1^* f_X\mathscr{F}, \overline{c}_2^! f_X\mathscr{F})) & \xrightarrow[\mathrm{tr}_{\overline{c}}]{} & \mathrm{H}^0(\overline{C}, \overline{\Delta}'_* K_{\mathrm{Fix}(\overline{c})}) \end{array} \tag{3.21}$$

The upper square commutes by the naturality of h^0_C, while the lower square commutes by the last Proposition 3.17. Hence, to prove the corollary, it suffices to prove that the vertical maps of (3.21) may be identified with L_F and R_F, respectively, i. e. that the two diagrams (3.22) and (3.23) below commute:

$$\begin{array}{ccccc} \mathrm{Hom}(c_{2!}c_1^*\mathscr{F},\mathscr{F}) & \xrightarrow{\cong} & \mathrm{Hom}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) & \xrightarrow{\cong} & \mathrm{H}^0(C, \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F})) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r_C\circ h^0_C} \\ \mathrm{Hom}(fc_{2!}c_1^*\mathscr{F}, f\mathscr{F}) & & \mathrm{Hom}(fc_1^*\mathscr{F}, fc_2^!\mathscr{F}) & \xrightarrow[\cong]{} & \mathrm{H}^0(\overline{C}, \mathbf{R}\underline{\mathrm{Hom}}(fc_1^*\mathscr{F}, fc_2^!\mathscr{F})) \\ \downarrow{\scriptstyle \mathrm{bc}^*_{c_2}\circ \mathrm{bc}_{!c_2}} & & \downarrow{\scriptstyle \mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2}} & & \downarrow{\scriptstyle \mathrm{bc}^*_{c_1}\times \mathrm{bc}^!_{c_2}} \\ \mathrm{Hom}(\overline{c}_{2!}\overline{c}_1^* f\mathscr{F}, f\mathscr{F}) & \xrightarrow[\cong]{} & \mathrm{Hom}(\overline{c}_1^* f\mathscr{F}, \overline{c}_2^! f\mathscr{F}) & \xrightarrow[\cong]{} & \mathrm{H}^0(\overline{C}, \mathbf{R}\underline{\mathrm{Hom}}(\overline{c}_1^* f\mathscr{F}, \overline{c}_2^! f\mathscr{F})) \end{array} \tag{3.22}$$

The upper right square commutes by 3.14.2, the lower right square commutes by the naturality of the horizontal arrows, while the commutativity of the left square follows easily from the definition of

$\mathrm{bc}_{!c_2}$.

$$\begin{array}{ccc} \mathrm{H}^0(C, \Delta'_* K_{\mathrm{Fix}(c)}) & \xrightarrow{\cong} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle \mathrm{bc}_{*\Delta'} \circ h^0_C} & & \downarrow{\scriptstyle h^0_{\mathrm{Fix}(c)}} \\ \mathrm{H}^0(\overline{C}, \overline{\Delta}'_* f K_{\mathrm{Fix}(c)}) & \xrightarrow{\cong} & \mathrm{H}^0(\mathrm{Fix}(\overline{c}), f K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}} & & \downarrow{\scriptstyle \tilde{\pi}_{\mathrm{Fix}(c)}} \\ \mathrm{H}^0(\overline{C}, \overline{\Delta}'_* K_{\mathrm{Fix}(\overline{c})}) & \xrightarrow{\cong} & \mathrm{H}^0(\mathrm{Fix}(\overline{c}), K_{\mathrm{Fix}(\overline{c})}) \end{array} \tag{3.23}$$

Here, the lower half commutes by the naturality of the horizontal arrows, while the upper half is commutative by 3.15.1. □

§5 Applications

We come now back to the questions raised at the end of §1, and propose to answer them by interpreting the result 3.18 of the last paragraph in the contexts considered in §1 (and in 3.6).

The following statement expresses the naturality of the trace map with respect to proper push-forward.

Corollary 3.19 *Let* $[f] = (f, f^\natural, f) : c \to \overline{c}$ *be a morphism in* **PCor**. *Then the following identities hold:*

$$L_{P([f])} = [f]_!, \qquad R_{P([f])} = \int_{f'}, \tag{3.24}$$

where $f' = f^\natural|_{\mathrm{Fix}(c)} : \mathrm{Fix}(c) \to \mathrm{Fix}(\overline{c})$. *In particular, the following square commutes for any* $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ *(if* $c : C \to X \times X$*):*

$$\begin{array}{ccc} \mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) & \xrightarrow{\mathrm{tr}_c} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\ \downarrow{\scriptstyle [f]_!} & & \downarrow{\scriptstyle \int_{f'}} \\ \mathrm{Hom}_{\overline{c}}(f_!\mathscr{F}, f_!\mathscr{F}) & \xrightarrow[\mathrm{tr}_{\overline{c}}]{} & \mathrm{H}^0(\mathrm{Fix}(\overline{c}), K_{\mathrm{Fix}(\overline{c})}) \end{array}$$

Proof In view of 3.18 and 3.9 it suffices to prove the identities (3.24). The first identity $L_{P([f])} = [f]_!$ is obvious by the definition of the two maps. For the second identity, let $u \in \mathrm{Hom}(\Lambda_{\mathrm{Fix}(c)}, K_{\mathrm{Fix}(c)})$. We will prove $R_{P([f])}(u) = \int_{f'} u$:

$$\begin{array}{ccccccc} \pi^*_{\mathrm{Fix}(\overline{c})} \mathbb{1}_* \Lambda_k & \xrightarrow{\mathrm{bc}\ (1.12)} & f'_! \pi^*_{\mathrm{Fix}(c)} \Lambda_k & \xrightarrow{f'_! u} & f'_! \pi^!_{\mathrm{Fix}(c)} \Lambda_k & \xrightarrow{\mathrm{bc}\ (1.11)} & \pi^!_{\mathrm{Fix}(\overline{c})} \Lambda_k \\ \| & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} & & \| \\ \pi^*_{\mathrm{Fix}(\overline{c})} \Lambda_k & \xrightarrow[\mathrm{adj}]{} & f'_! f'^* \pi^*_{\mathrm{Fix}(\overline{c})} \Lambda_k & \xrightarrow[f'_! u]{} & f'_! f'^! \pi^!_{\mathrm{Fix}(\overline{c})} \Lambda_k & \xrightarrow[\mathrm{adj}]{} & \pi^!_{\mathrm{Fix}(\overline{c})} \Lambda_k \end{array}$$

The top row is $R_{P([f])}(u)$, while the bottom row is $\int_{f'}(u)$. The middle square clearly commutes, while the outer squares commute because the base change morphisms are compatible with composition. □

We deduce the following well-known result.

Corollary 3.20 (Lefschetz-Verdier trace formula) *Let $c : C \to X \times X$ be a proper correspondence, $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$. Then we have the following equality:*

$$\mathrm{Tr}(\mathbf{R}\Gamma_c(u)) = \int_{\mathrm{Fix}(c)} \mathrm{tr}(u) = \sum_{\beta \in \pi_0(\mathrm{Fix}(c))} \mathrm{lt}_\beta(u),$$

where β runs over all connected components of $\mathrm{Fix}(c)$.

Proof By 3.2, the left hand side equals $\mathrm{tr}(\mathbf{R}\Gamma_c(u))$. And by 3.19, this is equal to $\mathrm{tr} \circ [\pi]_{c!}(u) = \int_{\mathrm{Fix}(c)} \mathrm{tr}(u)$, hence the first equality. For the second equality, denote by $j_\beta : \beta \hookrightarrow \mathrm{Fix}(c)$ the canonical inclusion of $\beta \in \pi_0(\mathrm{Fix}(c))$ and by $\mathrm{res}_\beta : \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \to \mathrm{H}^0(\beta, K_\beta)$ the canonical restriction morphism induced by $r_\beta : K_{\mathrm{Fix}(c)} \to j_{\beta *} j_\beta^* K_{\mathrm{Fix}(c)}$. Then we have $\int_{\pi_{\mathrm{Fix}(c)}} \int_{j_\beta} = \int_{\pi_\beta}$ since the composition of two adjunctions yields an adjunction. Hence the two inner squares of the following diagram commute:

$$\begin{array}{ccc}
\mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) & = & \mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) \\
\downarrow {\scriptstyle \oplus \mathrm{tr}_{c,\mathscr{F},\beta}} & & \downarrow {\scriptstyle \mathrm{tr}_{c,\mathscr{F}}} \\
\oplus_\beta \mathrm{H}^0(\beta, K_\beta) & \overset{\oplus \mathrm{res}_\beta}{\longleftarrow} \; \underset{\oplus \int_{j_\beta}}{\longrightarrow} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\
\downarrow {\scriptstyle \oplus \int_{\pi_\beta}} & & \downarrow {\scriptstyle \int_{\pi_{\mathrm{Fix}(c)}}} \\
\Lambda & = & \Lambda
\end{array}$$

By [2, XVII, 6.2.3], $\int_{j_\beta}$ is induced by $\mathrm{Tr}_{j_\beta} : j_{\beta !} j_\beta^* K_{\mathrm{Fix}(c)} \to K_{\mathrm{Fix}(c)}$ from which we deduce with [2, XVII, 6.2.3.1] that

$$\begin{aligned}
\mathbb{1}_{\mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)})} &= \mathrm{H}^0(\mathrm{Fix}(c), \mathrm{Tr}_{\mathbb{1}_{\mathrm{Fix}(c)}}) \\
&= \mathrm{H}^0(\mathrm{Fix}(c), \oplus_\beta \mathrm{Tr}_{j_\beta} \circ r_\beta) \\
&= \oplus_\beta \mathrm{H}^0(\mathrm{Fix}(c), \mathrm{Tr}_{j_\beta}) \circ \mathrm{H}^0(\mathrm{Fix}(c), r_\beta) \\
&= \oplus_\beta \int_{j_\beta} \mathrm{res}_\beta .
\end{aligned}$$

Hence:

$$\int_{\mathrm{Fix}(c)} \mathrm{tr} = \int_{\pi_{\mathrm{Fix}(c)}} \mathrm{tr}_{c,\mathscr{F}} = \int_{\pi_{\mathrm{Fix}(c)}} \left(\oplus_\beta \int_{j_\beta} \mathrm{res}_\beta \right) \mathrm{tr}_{c,\mathscr{F}} = \sum_\beta \int_{\pi_\beta} \mathrm{tr}_{c,\mathscr{F},\beta} = \sum_\beta \mathrm{lt}_\beta. \qquad \square$$

Next, we turn to specializations.

Corollary 3.21 *Let $\tilde{c} : c \to \tilde{c}_s$ be a morphism in* **SCor**. *Then the following identities hold:*

$$L_{S(\tilde{c})} = \mathrm{sp}_{\tilde{c}}, \qquad R_{S(\tilde{c})} = \mathrm{sp}_{\mathrm{Fix}(\tilde{c})}.$$

In particular, the following square commutes for any $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ (if $c : C \to X \times X$, $\tilde{c} : \tilde{C} \to \tilde{X} \times \tilde{X}$):

$$\begin{array}{ccc}
\mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) & \xrightarrow{\mathrm{tr}_c} & \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\
\downarrow {\scriptstyle \mathrm{sp}_{\tilde{c}}} & & \downarrow {\scriptstyle \mathrm{sp}_{\mathrm{Fix}(\tilde{c})}} \\
\mathrm{Hom}_{\tilde{c}_s}(\mathrm{sp}_{\tilde{X}} \mathscr{F}, \mathrm{sp}_{\tilde{X}} \mathscr{F}) & \xrightarrow[\mathrm{tr}_{\tilde{c}_s}]{} & \mathrm{H}^0(\mathrm{Fix}(\tilde{c}_s), K_{\mathrm{Fix}(\tilde{c}_s)})
\end{array}$$

Proof The first identity is clear. For the second identity, consider the following diagram, where the vertical morphisms are the canonical ones:

$$\begin{array}{ccccc}
\mathrm{H}^{\circ}(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) & \xrightarrow{h^{\circ}_{\mathrm{Fix}(c)}} & \mathrm{H}^{\circ}(\mathrm{Fix}(\tilde{c}_s), \mathrm{sp}_{\mathrm{Fix}(\tilde{c})} K_{\mathrm{Fix}(c)}) & \xrightarrow{\tilde{\pi}_{\mathrm{Fix}(c)}} & \mathrm{H}^{\circ}(\mathrm{Fix}(\tilde{c}_s), K_{\mathrm{Fix}(\tilde{c}_s)}) \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\
\mathrm{H}^{\circ}(k, \pi_{\mathrm{Fix}(c)*} K_{\mathrm{Fix}(c)}) & \xrightarrow[\mathrm{bc}_* \circ (2.10)]{} & \mathrm{H}^{\circ}(k, \varphi_{s*} \mathrm{sp}_{\mathrm{Fix}(\tilde{c})} K_{\mathrm{Fix}(c)}) & \xrightarrow[(2.10)^{-1} \circ \mathrm{bc}^!]{} & \mathrm{H}^{\circ}(k, \varphi_{s*} K_{\mathrm{Fix}(\tilde{c}_s)})
\end{array}$$

Here, $\varphi : \mathrm{Fix}(\tilde{c}) \to R$ denotes the structure morphism of $\mathrm{Fix}(\tilde{c})$, thus the bottom row is $\mathrm{sp}_{\mathrm{Fix}(\tilde{c})}$ (cf. 2.23). Since the top row is $R_{S(\tilde{c})}$ and since the right square obviously commutes, we are left to prove the commutativity of the left square.

Let $u \in \mathrm{Hom}(\Lambda_{\mathrm{Fix}(c)}, K_{\mathrm{Fix}(c)})$. Then $h^{\circ}_{\mathrm{Fix}(c)}(u)$ and $\mathrm{bc}_* \circ (2.10)(u)$ are the two outer paths $\Lambda_k \to \varphi_{s*} \mathrm{sp}_{\mathrm{Fix}(\tilde{c})} K_{\mathrm{Fix}(c)}$ in the following diagram:

$$\begin{array}{ccccccc}
\Lambda_k & \xrightarrow{(2.10) \circ \mathrm{adj}} & \varphi_{s*} \varphi_s^* \mathrm{sp}_R \Lambda_k & \xrightarrow{\varepsilon \circ \mathrm{bc}^*} & \varphi_{s*} \mathrm{sp}_{\mathrm{Fix}(\tilde{c})} \Lambda_{\mathrm{Fix}(c)} & \xrightarrow{u} & \varphi_{s*} \mathrm{sp}_{\mathrm{Fix}(\tilde{c})} K_{\mathrm{Fix}(c)} \\
\downarrow{\mathrm{adj}} & & & \nearrow{\scriptstyle \mathrm{bc}_* \circ (2.10)} & & \nearrow{\scriptstyle \mathrm{bc}_* \circ (2.10)} & \\
\pi_{\mathrm{Fix}(c)*} \pi^*_{\mathrm{Fix}(c)} \Lambda_k & \xrightarrow[\varepsilon]{\cong} & \pi_{\mathrm{Fix}(c)*} \Lambda_{\mathrm{Fix}(c)} & \xrightarrow[u]{} & \pi_{\mathrm{Fix}(c)*} K_{\mathrm{Fix}(c)} & &
\end{array}$$

The parallelogram on the right is commutative by the naturality of the slanted arrows. Expressing bc_* in terms of bc^* (by adjointness, cf. page 23) renders the proof of the commutativity of the trapezoid on the left an easy matter. The second statement of the corollary follows from 3.18 and 3.9. □

Finally, we may also prove that the trace map is natural with respect to restriction.

Corollary 3.22 *Let $c : C \to X \times X$ be a correspondence, let $W \hookrightarrow C$ be an open subset, and set $\beta = W \cap \mathrm{Fix}(c)$. Then the following identities hold:*

$$L_{Q([j_W])} = [j_W]^*, \qquad R_{Q([j_W])} = \mathrm{res}_{\beta}.$$

In particular, the following diagram commutes for every $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$:

$$\begin{array}{ccc}
\mathrm{Hom}_c(\mathscr{F}, \mathscr{F}) & \xrightarrow{\mathrm{tr}} & \mathrm{H}^{\circ}(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \\
\downarrow{[j_W]^*} & & \downarrow{\mathrm{res}_{\beta}} \\
\mathrm{Hom}_{c|_W}(\mathscr{F}, \mathscr{F}) & \xrightarrow[\mathrm{tr}]{} & \mathrm{H}^{\circ}(\beta, K_{\beta})
\end{array}$$

Proof As in 3.19, the identities are easily checked, and the second statement then follows from 3.18 and 3.9. □

§6 Additivity

Let $c : C \to X \times X$ be a correspondence, $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$, $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$ a cohomological correspondence and $i : Z \hookrightarrow X$ a closed subscheme such that $c|^Z$ (and hence $u|^Z = [i^Z]^*(u)$) exists (2.4.3, 2.13.3). Then $[i^Z] : c|^Z \hookrightarrow c$ is a closed immersion and satisfies condition (F2) of 2.§3. Hence we may define the pushforward

$$[i^Z]_!(u|^Z) \in \mathrm{Hom}_c(i_! i^* \mathscr{F}, i_! i^* \mathscr{F}).$$

Similarly, let $U = X\backslash Z$, $j : U \hookrightarrow X$ the inclusion. In 2.13.2 we defined the pullback $[j^U]^*(u)$ with respect to the open immersion $[j^U] : c|^U \hookrightarrow c$. Now, since $c_1(c_2^{-1}(Z)) \subset Z$ set-theoretically we also have $c_1^{-1}(U) \subset C\backslash c_2^{-1}(Z) = c_2^{-1}(U)$ hence condition (F1) of 2.§3 is satisfied and we may define the pushforward

$$[j^U]_!(u|^U) \in \mathrm{Hom}_c(j_! j^* \mathscr{F}, j_! j^* \mathscr{F}).$$

Set $\mathscr{F}_Z := i_! i^* \mathscr{F}$ and $\mathscr{F}_U := j_! j^* \mathscr{F}$. The result to be proved in this paragraph gives a simple relationship between the traces associated to the different cohomological correspondences appearing above:

Proposition 3.23 *In the notation just introduced, the following identity holds in* $\mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)})$*:*

$$\mathrm{tr}_{\mathscr{F}}(u) = \mathrm{tr}_{\mathscr{F}_Z}([i^Z]_!(u|^Z)) + \mathrm{tr}_{\mathscr{F}_U}([j^U]_!(u|^U)).$$

We will deduce this identity from the additivity of filtered trace maps. Recall the notation and definitions of 1.§5. In particular, fix abelian categories **A**, **B**, and **C** as there.

The next two lemmas provide a means to transfer properties of functors and natural transformations from the derived categories to their filtered counterparts.

Lemma 3.24

1. *Let* $G : \mathfrak{D}(\mathbf{A}) \to \mathfrak{D}(\mathbf{B})$ *(resp.* $\mathfrak{D}(\mathbf{A})^\circ \to \mathfrak{D}(\mathbf{B})$*) be a functor and* $(\tilde{G}, \varphi_G)$ *a filtered lift of* G*. Then there exists, for every* $i \in \mathbb{Z}$*, a natural isomorphism of triangulated functors* $\chi^i : \mathrm{gr}^i \tilde{G} \to G\mathrm{gr}^i$ *(resp.* $\chi^i : \mathrm{gr}^i \tilde{G} \to G\mathrm{gr}^{-i}$*) such that restricted to* $\mathfrak{Df}^{[i,i]}(\mathbf{A})$ *(resp.* $\mathfrak{Df}^{[-i,-i]}(\mathbf{A})$*) it is canonically identified with* φ_G*.*
2. *Similarly, let* $G : \mathfrak{D}(\mathbf{A}) \times \mathfrak{D}(\mathbf{B}) \to \mathfrak{D}(\mathbf{C})$ *(resp.* $\mathfrak{D}(\mathbf{A})^\circ \times \mathfrak{D}(\mathbf{B}) \to \mathfrak{D}(\mathbf{C})$*) be a bifunctor and* $(\tilde{G}, \varphi_G)$ *a filtered lift of* G*. Then there exists, for every* $i \in \mathbb{Z}$*, a natural isomorphism* $\chi^i : \mathrm{gr}^i \tilde{G} \to \oplus_{r+s=i} G(\mathrm{gr}^r \times \mathrm{gr}^s)$ *(resp.* $\chi^i : \mathrm{gr}^i \tilde{G} \to \oplus_{r+s=i} G(\mathrm{gr}^{-r} \times \mathrm{gr}^s)$*) such that restricted to* $\mathfrak{Df}^{[r,r]}(\mathbf{A}) \times \mathfrak{Df}^{[s,s]}(\mathbf{B})$ *(resp.* $\mathfrak{Df}^{[-r,-r]}(\mathbf{A}) \times \mathfrak{Df}^{[s,s]}(\mathbf{B})$*) with* $r+s=i$ *it is canonically identified with* φ_G*.*

More explicitly, this identification is given as follows (in the case of a covariant bifunctor): Since $\tilde{G}$ is filtered it takes $\mathfrak{Df}^{[r,r]}(\mathbf{A}) \times \mathfrak{Df}^{[s,s]}(\mathbf{B})$ to $\mathfrak{Df}^{[i,i]}(\mathbf{C})$. Moreover, the functors $\tau^{[a,a]}$ are naturally isomorphic to the identity functors on $\mathfrak{Df}^{[a,a]}$ hence χ^i is naturally identified with a transformation $\omega\tilde{G} \to G(\omega \times \omega)$ and we require this transformation to be φ_G.

Proof We will do only part 2 in the case of a covariant bifunctor (all four cases are obviously similar). Fix $i, r, s \in \mathbb{Z}$ such that $r + s = i$. We first construct a natural isomorphism of functors

$$\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s}) \xrightarrow{\cong} G(\mathrm{gr}^r \times \mathrm{gr}^s). \tag{3.25}$$

Since $\tilde{G}$ is filtered, $\tilde{G}(\tau^{\le r} \times \tau^{\le s-1})$ is a functor with target $\mathfrak{Df}^{\le i-1}(\mathbf{C})$ hence $\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s-1})$ is the zero functor. Composing the distinguished triangle of functors $\tau^{\ge s} \to \mathbb{1} \to \tau^{\le s-1} \to^+$ (on $\mathfrak{Df}(\mathbf{B})$) with the functor $\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s})$ yields another distinguished triangle

$$\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{[s,s]}) \longrightarrow \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s}) \longrightarrow \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s-1}) \longrightarrow^+,$$

from which we deduce a natural isomorphism $\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{[s,s]}) \to \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s})$. A similar argument shows the existence of an isomorphism $\mathrm{gr}^i \tilde{G}(\tau^{[r,r]} \times \tau^{[s,s]}) \to \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{[s,s]})$. Composing these two isomorphisms we obtain

$$\mathrm{gr}^i \tilde{G}(\tau^{[r,r]} \times \tau^{[s,s]}) \xrightarrow{\cong} \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s}). \tag{3.26}$$

Now, (3.25) can be written as

$$\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s}) \xrightarrow{(3.26)^{-1}} \mathrm{gr}^i \tilde{G}(\tau^{[r,r]} \times \tau^{[s,s]}) \cong \omega \tilde{G}(\tau^{[r,r]} \times \tau^{[s,s]}) \xrightarrow{\varphi_G} G(\mathrm{gr}^r \times \mathrm{gr}^s).$$

Notice that restricted to $\mathfrak{Df}^{[r,r]}(\mathbf{A}) \times \mathfrak{Df}^{[s,s]}(\mathbf{B})$ this morphism is canonically identified with φ_G (in the sense made explicit before the proof).

Composing (3.25) with the canonical morphisms $\mathbb{1} \to \tau^{\le r}$ and $\mathbb{1} \to \tau^{\le s}$ we get a morphism of functors:

$$\mathrm{gr}^i \tilde{G} \longrightarrow \oplus_{r+s=i} \mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s}) \xrightarrow{(3.25)} \oplus_{r+s=i} G(\mathrm{gr}^r \times \mathrm{gr}^s). \tag{3.27}$$

(Notice that there are only finitely many pairs (r,s) such that $r+s=i$ and $\mathrm{gr}^i \tilde{G}(\tau^{\le r} \times \tau^{\le s})$ doesn't vanish hence the first morphism really maps into a direct sum.) Since (3.27) is a morphism of triangulated bifunctors, a standard argument in homological algebra shows that to prove it an isomorphism it suffices to consider the case where it is evaluated at $(L,M) \in \mathfrak{Df}^{[a,a]}(\mathbf{A}) \times \mathfrak{Df}^{[b,b]}(\mathbf{B})$, some $(a,b) \in \mathbb{Z}^2$. But then, both sides of (3.27) vanish unless $a+b=i$ in which case the morphism is canonically identified with φ_G as has already been observed. □

Remark 3.25 Assume for this remark that $\mathbf{A}$ has enough injectives thus $\widetilde{\mathbf{R}\mathrm{Hom}}$ is defined (cf. 1.(s)). Let $L \in \mathfrak{Df}(\mathbf{A})$, $M \in \mathfrak{Df}^+(\mathbf{A})$ and $u \in \mathrm{Hom}(L,M)$. It is then easy to see from the explicit description of χ^0 given in the lemma that the image of u under the map

$$\mathrm{Hom}(L,M) \xrightarrow{\cong} \mathrm{H}^0\, \omega\tau^{\ge 0}\widetilde{\mathbf{R}\mathrm{Hom}}(L,M) \xrightarrow{p} \mathrm{H}^0\, \mathrm{gr}^0 \widetilde{\mathbf{R}\mathrm{Hom}}(L,M) \xrightarrow[\cong]{\chi^0} \oplus_i \mathrm{Hom}(\mathrm{gr}^i L, \mathrm{gr}^i M)$$

is nothing but $\oplus_i \mathrm{gr}^i u$ (here p is induced by the projection $\omega\tau^{\ge 0} \to \mathrm{gr}^o$).

Lemma 3.26 *1. Let $H : G_1 \to G_2$ be a morphism of triangulated functors (resp. contravariant functors) and let $\tilde{H} : \tilde{G}_1 \to \tilde{G}_2$ be a filtered lift of H. Then the following diagram commutes for every $i \in \mathbb{Z}$:*

$$\begin{array}{ccc} \mathrm{gr}^i \tilde{G}_1 & \xrightarrow{\mathrm{gr}^i \tilde{H}} & \mathrm{gr}^i \tilde{G}_2 \\ \downarrow{\scriptstyle \chi^i} & & \downarrow{\scriptstyle \chi^i} \\ G_1 \mathrm{gr}^i & \xrightarrow[H\mathrm{gr}^i]{} & G_2 \mathrm{gr}^i \end{array} \qquad \textit{respectively} \qquad \begin{array}{ccc} \mathrm{gr}^i \tilde{G}_1 & \xrightarrow{\mathrm{gr}^i \tilde{H}} & \mathrm{gr}^i \tilde{G}_2 \\ \downarrow{\scriptstyle \chi^i} & & \downarrow{\scriptstyle \chi^i} \\ G_1 \mathrm{gr}^{-i} & \xrightarrow[H\mathrm{gr}^{-i}]{} & G_2 \mathrm{gr}^{-i} \end{array}$$

2. Let $H : G_1 \to G_2$ be a morphism of triangulated bifunctors (resp. bifunctors contravariant in the first argument) and let $\tilde{H} : \tilde{G}_1 \to \tilde{G}_2$ be a filtered lift of H. Then the following diagram commutes for every $i \in \mathbb{Z}$:

$$\begin{array}{ccc} \mathrm{gr}^i \tilde{G}_1 & \xrightarrow{\mathrm{gr}^i \tilde{H}} & \mathrm{gr}^i \tilde{G}_2 \\ \downarrow{\scriptstyle \chi^i} & & \downarrow{\scriptstyle \chi^i} \\ \oplus_{r+s=i} G_1(\mathrm{gr}^r \times \mathrm{gr}^s) & \xrightarrow[\oplus H(\mathrm{gr}^r \times \mathrm{gr}^s)]{} & \oplus_{r+s=i} G_2(\mathrm{gr}^r \times \mathrm{gr}^s) \end{array} \tag{3.28}$$

respectively

$$\begin{array}{ccc} \mathrm{gr}^i\tilde{G}_1 & \xrightarrow{\mathrm{gr}^i\tilde{H}} & \mathrm{gr}^i\tilde{G}_2 \\ {\scriptstyle\chi^i}\downarrow & & \downarrow{\scriptstyle\chi^i} \\ \oplus_{r+s=i} G_1(\mathrm{gr}^{-r}\times \mathrm{gr}^s) & \xrightarrow[\oplus H(\mathrm{gr}^{-r}\times\mathrm{gr}^s)]{} & \oplus_{r+s=i} G_2(\mathrm{gr}^{-r}\times \mathrm{gr}^s) \end{array} \tag{3.29}$$

3. *If H in 1 (resp. 2) is an isomorphism of functors (resp. bifunctors) then so is $\tilde{H}$.*

Proof We will again do only the case of a covariant bifunctor.

1, 2. Fix $i, r, s \in \mathbb{Z}$ such that $r + s = i$. In (3.28), we may compose the χ^i's with the projection onto the $(r, s)^{\text{th}}$ factor and thus obtain

$$\begin{array}{ccc} \mathrm{gr}^i\tilde{G}_1 & \xrightarrow{\mathrm{gr}^i\tilde{H}} & \mathrm{gr}^i\tilde{G}_2 \\ \downarrow & & \downarrow \\ G_1(\mathrm{gr}^{r}\times \mathrm{gr}^s) & \xrightarrow[H(\mathrm{gr}^{r}\times\mathrm{gr}^s)]{} & G_2(\mathrm{gr}^{r}\times \mathrm{gr}^s) \end{array}$$

It suffices to prove that this diagram commutes. According to the definition of χ^i, it is to be expanded as follows:

$$\begin{array}{ccc} \mathrm{gr}^i\tilde{G}_1 & \xrightarrow{\mathrm{gr}^i\tilde{H}} & \mathrm{gr}^i\tilde{G}_2 \\ \downarrow & & \downarrow \\ \mathrm{gr}^i\tilde{G}_1(\tau^{\leq r}\times\tau^{\leq s}) & \xrightarrow{\mathrm{gr}^i\tilde{H}(\tau^{\leq r}\times\tau^{\leq s})} & \mathrm{gr}^i\tilde{G}_2(\tau^{\leq r}\times\tau^{\leq s}) \\ {\scriptstyle(3.26)}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle(3.26)} \\ \mathrm{gr}^i\tilde{G}_1(\tau^{[r,r]}\times\tau^{[s,s]}) & \xrightarrow{\mathrm{gr}^i\tilde{H}(\tau^{[r,r]}\times\tau^{[s,s]})} & \mathrm{gr}^i\tilde{G}_2(\tau^{[r,r]}\times\tau^{[s,s]}) \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\ \omega\tilde{G}_1(\tau^{[r,r]}\times\tau^{[s,s]}) & \xrightarrow{\omega\tilde{H}(\tau^{[r,r]}\times\tau^{[s,s]})} & \omega\tilde{G}_2(\tau^{[r,r]}\times\tau^{[s,s]}) \\ {\scriptstyle\varphi_{G_1}}\downarrow & & \downarrow{\scriptstyle\varphi_{G_2}} \\ G_1(\mathrm{gr}^{r}\times \mathrm{gr}^s) & \xrightarrow[H(\mathrm{gr}^{r}\times\mathrm{gr}^s)]{} & G_2(\mathrm{gr}^{r}\times \mathrm{gr}^s) \end{array}$$

The first two squares commute since $\mathrm{gr}^i\tilde{H}$ is a natural transformation, the third square clearly commutes, while the last square commutes by the definition of a filtered lift of a natural transformation. Thus the whole diagram is commutative.

3. Since $\tilde{H}$ is a morphism of triangulated bifunctors, it suffices, as before, to prove that it is an isomorphism when restricted to $\mathfrak{Df}^{[a,a]}(\mathbf{A}) \times \mathfrak{Df}^{[b,b]}(\mathbf{B})$ for all $a, b \in \mathbb{Z}$ hence it suffices to prove that $\mathrm{gr}^i\tilde{H}$ is an isomorphism on $\mathfrak{Df}^{[a,a]}(\mathbf{A}) \times \mathfrak{Df}^{[b,b]}(\mathbf{B})$ for $i = a + b$, and this is, by 2, equivalent to $\oplus_{r+s=i} H(\mathrm{gr}^r \times \mathrm{gr}^s)$ being an isomorphism on the same subcategory. And this last condition is obviously satisfied. □

Definition 3.27 Let X be a scheme. We define $\mathfrak{D}^b_{ctf}\mathfrak{f}(X)$ to be the full subcategory of $\mathfrak{D}^b\mathfrak{f}(X)$ consisting of those objects $\tilde{\mathscr{F}}$ such that $\mathrm{gr}^i\tilde{\mathscr{F}}$ belongs to $\mathfrak{D}^b_{ctf}(X)$ for all $i \in \mathbb{Z}$ (see 1.(s) for the notation).

Our next task is to define the six operations in the filtered context.

Lemma 3.28 *Let $f : X \to Y$ be a morphism of schemes. The filtered lifts of f_*, f^*, $\mathbf{R}\underline{\mathrm{Hom}}$, $\otimes$ (cf. 1.(s)) define (bi)functors*

$$\begin{aligned}\tilde{f}_* &: \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \longrightarrow \mathfrak{D}^b_{ctf}\mathfrak{f}(Y),\\ \tilde{f}^* &: \mathfrak{D}^b_{ctf}\mathfrak{f}(Y) \longrightarrow \mathfrak{D}^b_{ctf}\mathfrak{f}(X),\\ \widetilde{\mathbf{R}\underline{\mathrm{Hom}}} &: \mathfrak{D}^b_{ctf}\mathfrak{f}(X)^\circ \times \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \longrightarrow \mathfrak{D}^b_{ctf}\mathfrak{f}(X),\\ \widetilde{\otimes} &: \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \times \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \longrightarrow \mathfrak{D}^b_{ctf}\mathfrak{f}(X).\end{aligned}$$

Proof Let $\tilde{\mathscr{F}} \in \mathfrak{D}^b_{ctf}\mathfrak{f}(X)$. By 3.24 we then have for any $i \in \mathbb{Z}$,

$$\mathrm{gr}^i\tilde{f}_*(\tilde{\mathscr{F}}) \cong f_*\mathrm{gr}^i(\tilde{\mathscr{F}}) \in f_*\mathrm{gr}^i(\mathfrak{D}^b_{ctf}\mathfrak{f}(X)) \subset f_*(\mathfrak{D}^b_{ctf}(X)) \subset \mathfrak{D}^b_{ctf}(Y)$$

which proves the claim for $\tilde{f}_*$ (since, by the same isomorphism, $\tilde{f}_*(\tilde{\mathscr{F}})$ belongs to $\mathfrak{D}^b\mathfrak{f}(Y)$). The argument for the other functors is the same. □

Definition 3.29 1. Let X be a scheme and denote by $\tilde{K}_X$ the complex K_X considered as an object of $\mathfrak{D}^b_{ctf}\mathfrak{f}^{[0,0]}(X)$. We define the *filtered Verdier duality functor*:

$$\tilde{\mathbb{D}} = \tilde{\mathbb{D}}_X = \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(-, \tilde{K}_X) : \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \to \mathfrak{D}^b_{ctf}\mathfrak{f}(X).$$

2. Let $f : X \to Y$ be a morphism of schemes. We define two functors:

$$\begin{aligned}\tilde{f}_! &= \tilde{\mathbb{D}}_Y\tilde{f}_*\tilde{\mathbb{D}}_X : \mathfrak{D}^b_{ctf}\mathfrak{f}(X) \to \mathfrak{D}^b_{ctf}\mathfrak{f}(Y),\\ \tilde{f}^! &= \tilde{\mathbb{D}}_X\tilde{f}^*\tilde{\mathbb{D}}_Y : \mathfrak{D}^b_{ctf}\mathfrak{f}(Y) \to \mathfrak{D}^b_{ctf}\mathfrak{f}(X).\end{aligned}$$

Lemma 3.30 1. *The functors $\tilde{\mathbb{D}}$, $\tilde{f}_!$ and $\tilde{f}^!$ are filtered lifts of $\mathbb{D}$, $f_!$ and $f^!$, respectively.*

2. *The natural isomorphism $\mathbb{D}^2 \to \mathbb{1}$ lifts to an isomorphism of filtered functors $\tilde{\mathbb{D}}^2 \to \mathbb{1}$.*

3. *There is a natural isomorphism of filtered bifunctors*

$$\tilde{f}_*\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(-, \tilde{f}^!-) \cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{f}_!-, -).$$

In particular, the adjoint relation $\tilde{f}_! \dashv \tilde{f}^!$ still holds in the filtered context.

Proof 1. Being defined as a composition of filtered functors all three functors are clearly filtered. Moreover,

$$\omega\tilde{\mathbb{D}} = \omega\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(-, \tilde{K}_X) \cong \mathbf{R}\underline{\mathrm{Hom}}(\omega-, K_X) = \mathbb{D}\omega,$$

hence $\tilde{\mathbb{D}}$ is a lift of $\mathbb{D}$. That $\tilde{f}^!$ and $\tilde{f}_!$ lift the functors $f^!$ and $f_!$, respectively, follows in the same manner starting from the description of these functors as "dual" to f^* and f_*, respectively (cf. [14, I, 1.12]).

2. The isomorphism $\mathbb{D}^2 \to \mathbb{1}$ is defined in terms of the adjunction between $\otimes$ and $\mathbf{R}\underline{\mathrm{Hom}}$ (cf. [14, I, p. 7–8]) thus it has a filtered lift which is, by 3.26.3, an isomorphism.

3. Let $f : X \to Y$ be our morphism of schemes and fix $\tilde{\mathscr{F}}, \tilde{\mathscr{G}} \in \mathfrak{D}^b_{ctf}\mathfrak{f}(X)$, $\tilde{\mathscr{H}} \in \mathfrak{D}^b_{ctf}\mathfrak{f}(Y)$. Then there exists an isomorphism

$$\tilde{f}_* \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{f}^* \tilde{\mathscr{H}}, \tilde{\mathscr{F}}) \cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{H}}, \tilde{f}_* \tilde{\mathscr{F}}), \tag{3.30}$$

natural in both $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{H}}$ (see [15, V, 2.3.5.6]). Next, there is an isomorphism

$$\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}}, \tilde{\mathbb{D}}\tilde{\mathscr{G}}) \cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{G}}, \tilde{\mathbb{D}}\tilde{\mathscr{F}}), \tag{3.31}$$

also natural in both $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$. Indeed, the latter is obtained as the composition of the natural isomorphisms

$$\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}}, \tilde{\mathbb{D}}\tilde{\mathscr{G}}) \cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}} \tilde{\otimes} \tilde{\mathscr{G}}, \tilde{K}_X) \cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{G}}, \tilde{\mathbb{D}}\tilde{\mathscr{F}}),$$

from [15, V, 2.3.1.3].

Now, it's simply a question of composing these isomorphisms in the correct order:

$$\begin{aligned}
\tilde{f}_* \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}}, \tilde{f}^! \tilde{\mathscr{H}}) &= \tilde{f}_* \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}}, \tilde{\mathbb{D}} \tilde{f}^* \tilde{\mathbb{D}} \tilde{\mathscr{H}}) && \\
&\cong \tilde{f}_* \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{f}^* \tilde{\mathbb{D}} \tilde{\mathscr{H}}, \tilde{\mathbb{D}}\tilde{\mathscr{F}}) && (3.31) \\
&\cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathbb{D}} \tilde{\mathscr{H}}, \tilde{f}_* \tilde{\mathbb{D}}\tilde{\mathscr{F}}) && (3.30) \\
&\cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathbb{D}} \tilde{\mathscr{H}}, \tilde{\mathbb{D}}^2 \tilde{f}_* \tilde{\mathbb{D}}\tilde{\mathscr{F}}) && \text{by part 2} \\
&\cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathbb{D}} \tilde{f}_* \tilde{\mathbb{D}}\tilde{\mathscr{F}}, \tilde{\mathbb{D}}^2 \tilde{\mathscr{H}}) && (3.31) \\
&\cong \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathbb{D}} \tilde{f}_* \tilde{\mathbb{D}}\tilde{\mathscr{F}}, \tilde{\mathscr{H}}) && \text{by part 2} \\
&= \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{f}_! \tilde{\mathscr{F}}, \tilde{\mathscr{H}}). &&
\end{aligned}$$

The last statement of the lemma is obtained by applying the functor $\mathrm{H}^0(X, \omega\tau^{\geq 0})$ to this isomorphism. □

Fix a correspondence $c : C \to X \times X$, $\tilde{\mathscr{F}} \in \mathfrak{D}^b_{ctf}\mathfrak{f}(X)$ and a cohomological correspondence $u \in \mathrm{Hom}(\tilde{c}_{2!}\tilde{c}_1^* \tilde{\mathscr{F}}, \tilde{\mathscr{F}})$. We abbreviate $\mathscr{F} := \omega\tilde{\mathscr{F}}$. In view of the isomorphisms

$$\omega \tilde{c}_{2!}\tilde{c}_1^* \tilde{\mathscr{F}} \cong c_{2!} c_1^* \mathscr{F}, \qquad \mathrm{gr}^i \tilde{c}_{2!}\tilde{c}_1^* \tilde{\mathscr{F}} \cong c_{2!} c_1^* \mathrm{gr}^i \tilde{\mathscr{F}}$$

(the second one existing by 3.24.1), we may consider ωu and $\mathrm{gr}^i u$ as elements of $\mathrm{Hom}(c_{2!}c_1^*\mathscr{F}, \mathscr{F})$ and $\mathrm{Hom}(c_{2!}c_1^* \mathrm{gr}^i \tilde{\mathscr{F}}, \mathrm{gr}^i \tilde{\mathscr{F}})$, respectively (any $i \in \mathbb{Z}$). The next proposition expresses the additivity of filtered trace maps.

Proposition 3.31 *With the identifications just explained, we have an equality*

$$\mathrm{tr}_{\mathscr{F}}(\omega u) = \sum_{i \in \mathbb{Z}} \mathrm{tr}_{\mathrm{gr}^i \tilde{\mathscr{F}}}(\mathrm{gr}^i u) \tag{3.32}$$

in $\mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)})$.

The proof will be divided into three steps.

Step 1 *The global trace morphism associated to $\mathscr{F}$ possesses a natural filtered lift $\underline{\tilde{\mathrm{tr}}}_{\tilde{\mathscr{F}}}$ in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) & \xrightarrow{\underline{\mathrm{tr}}_{\mathscr{F}}} & \Delta'_* K_{\mathrm{Fix}(c)} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \omega\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_1^*\tilde{\mathscr{F}}, \tilde{c}_2^!\tilde{\mathscr{F}}) & \xrightarrow[\omega\underline{\tilde{\mathrm{tr}}}_{\tilde{\mathscr{F}}}]{} & \omega\tilde{\Delta}'_*\tilde{K}_{\mathrm{Fix}(c)} \end{array}$$

Proof Indeed, the global trace morphism is made up of adjunctions between the six operations, of canonical (iso)morphisms coming from the level of complexes and inverted such isomorphisms. As we saw, the adjunction relations still hold in the filtered context (the units and counits lifting their unfiltered counterpart), and the other canonical morphisms can be directly seen to preserve the filtrations (see e. g. [15, V, 2.3.2.3, 2.3.4.3, 2.3.5.5]). Finally, 3.26.3 shows that the filtered lifts of isomorphisms are still isomorphisms and may thus be inverted in the filtered context as well. □

Composing this filtered trace morphism with the isomorphism from 3.30.3 yields a map

$$\underline{\tilde{\mathrm{tr}}}'_{\tilde{\mathscr{F}}} : \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}}, \tilde{\mathscr{F}}) \longrightarrow \tilde{c}'_*\tilde{K}_{\mathrm{Fix}(c)}$$

lifting the corresponding composition in the unfiltered context (again, in the sense of a commutative diagram as above),

$$\underline{\mathrm{tr}}'_{\mathscr{F}} : \mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathscr{F}, \mathscr{F}) \longrightarrow c_{2*}\,\mathbf{R}\underline{\mathrm{Hom}}(c_1^*\mathscr{F}, c_2^!\mathscr{F}) \xrightarrow{c_{2*}\underline{\mathrm{tr}}_{\mathscr{F}}} c'_*K_{\mathrm{Fix}(c)}.$$

Step 2 *The following diagram commutes for any $i \in \mathbb{Z}$:*

$$\begin{array}{ccc} \mathrm{gr}^\circ\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}}, \tilde{\mathscr{F}}) & \xrightarrow{\mathrm{gr}^\circ\underline{\tilde{\mathrm{tr}}}'_{\tilde{\mathscr{F}}}} & \omega\tilde{c}'_*\tilde{K}_{\mathrm{Fix}(c)} \\ \uparrow{\scriptstyle\cong} & & \| \\ \oplus_i\,\mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathrm{gr}^i\tilde{\mathscr{F}}, \mathrm{gr}^i\tilde{\mathscr{F}}) & & \| \\ \uparrow & & \| \\ \mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathrm{gr}^i\tilde{\mathscr{F}}, \mathrm{gr}^i\tilde{\mathscr{F}}) & \xrightarrow[\underline{\mathrm{tr}}'_{\mathrm{gr}^i\tilde{\mathscr{F}}}]{} & c'_*K_{\mathrm{Fix}(c)} \end{array}$$

Proof We decompose the trace morphisms as follows:

$$\begin{array}{ccccc} \mathrm{gr}^\circ\widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}}, \tilde{\mathscr{F}}) & \xrightarrow{\mathrm{gr}^\circ\underline{\tilde{\mathrm{tr}}}''_{\tilde{\mathscr{F}}}} & \mathrm{gr}^\circ\tilde{c}'_*\tilde{c}'^!(\tilde{\mathbb{D}}\tilde{\mathscr{F}}\tilde{\otimes}\tilde{\mathscr{F}}) & \xrightarrow{\mathrm{gr}^\circ\tilde{c}'_*\tilde{c}'^!\tilde{\mathrm{ev}}_{\tilde{\mathscr{F}}}} & \omega\tilde{c}'_*\tilde{K}_{\mathrm{Fix}(c)} \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \| \\ \oplus_i\,\mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathrm{gr}^i\tilde{\mathscr{F}}, \mathrm{gr}^i\tilde{\mathscr{F}}) & & \oplus_i c'_*c'^!(\mathbb{D}\mathrm{gr}^i\tilde{\mathscr{F}}\otimes\mathrm{gr}^i\tilde{\mathscr{F}}) & \xrightarrow{c'_*c'^!\mathrm{gr}^\circ\tilde{\mathrm{ev}}_{\tilde{\mathscr{F}}}} & c'_*K_{\mathrm{Fix}(c)} \\ \uparrow & & \uparrow & & \| \\ \mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathrm{gr}^i\tilde{\mathscr{F}}, \mathrm{gr}^i\tilde{\mathscr{F}}) & \xrightarrow[\underline{\mathrm{tr}}''_{\mathrm{gr}^i\tilde{\mathscr{F}}}]{} & c'_*c'^!(\mathbb{D}\mathrm{gr}^i\tilde{\mathscr{F}}\otimes\mathrm{gr}^i\tilde{\mathscr{F}}) & \xrightarrow[c'_*c'^!\mathrm{ev}_{\mathrm{gr}^i\tilde{\mathscr{F}}}]{} & c'_*K_{\mathrm{Fix}(c)} \end{array}$$

The left inner square commutes by 3.26.2, the top right inner square does so by 3.26.1.

To check the commutativity of the bottom right inner square, consider the following isomorphism of bifunctors

$$\varphi_{\mathscr{A},\mathscr{B}} : \mathbf{R}\underline{\mathrm{Hom}}(\mathbb{D}\mathscr{A} \otimes \mathscr{B}, K_X) \xrightarrow{\cong} \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{B}, \mathbb{D}^2\mathscr{A}) \xrightarrow{\cong} \mathbf{R}\underline{\mathrm{Hom}}(\mathscr{B}, \mathscr{A}).$$

The first arrow comes from an adjunction thus has a filtered lift, the second one has a filtered lift by 3.30.2. By 3.26 then, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{gr}^0 \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathbb{D}}\tilde{\mathscr{F}} \tilde{\otimes} \tilde{\mathscr{F}}, \tilde{K}_X) & \xrightarrow[\cong]{\mathrm{gr}^0 \tilde{\varphi}} & \mathrm{gr}^0 \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{\mathscr{F}}, \tilde{\mathscr{F}}) \\
\cong \downarrow & & \downarrow \cong \\
\oplus_i \mathbf{R}\underline{\mathrm{Hom}}(\mathbb{D}\mathrm{gr}^i \tilde{\mathscr{F}} \otimes \mathrm{gr}^i \tilde{\mathscr{F}}, K_X) & \xrightarrow[\cong]{\oplus_i \varphi} & \oplus_i \mathbf{R}\underline{\mathrm{Hom}}(\mathrm{gr}^i \tilde{\mathscr{F}}, \mathrm{gr}^i \tilde{\mathscr{F}})
\end{array}$$

The claim now follows from the fact that $\mathrm{gr}^0 \tilde{\varphi}(\mathrm{gr}^0 \tilde{\mathrm{ev}}) = \mathrm{gr}^0 \mathbb{1}$ is mapped to $\oplus_i \varphi(\mathrm{gr}^i \tilde{\mathrm{ev}}) = \oplus_i \mathbb{1}$ under the vertical isomorphism. □

Step 3 *End of the proof of 3.31.*

Proof Denote by p the projection $\omega\tau^{\geq 0} \to \mathrm{gr}^0$. Then the following diagram clearly commutes:

$$\begin{array}{ccc}
\omega\tau^{\geq 0} \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_{2!}\tilde{c}_1^* \tilde{\mathscr{F}}, \tilde{\mathscr{F}}) & \xrightarrow{\omega\tau^{\geq 0} \underline{\tilde{\mathrm{tr}}}_{\tilde{\mathscr{F}}}} & \omega \tilde{c}'_* \tilde{K}_{\mathrm{Fix}(c)} \\
p \downarrow & & \| \\
\mathrm{gr}^0 \widetilde{\mathbf{R}\underline{\mathrm{Hom}}}(\tilde{c}_{2!}\tilde{c}_1^* \tilde{\mathscr{F}}, \tilde{\mathscr{F}}) & \xrightarrow{\mathrm{gr}^0 \underline{\tilde{\mathrm{tr}}}_{\tilde{\mathscr{F}}}} & \omega \tilde{c}'_* \tilde{K}_{\mathrm{Fix}(c)}
\end{array} \tag{3.33}$$

Hence the left hand side of (3.32) is equal to

$$\begin{aligned}
\mathrm{tr}'_{\mathscr{F}}(\omega u) &= \mathrm{H}^0(X, \underline{\mathrm{tr}}'_{\mathscr{F}}(\omega u)) \\
&= \mathrm{H}^0(X, \omega\tau^{\geq 0} \underline{\tilde{\mathrm{tr}}}'_{\tilde{\mathscr{F}}}(u)) \\
&= \mathrm{H}^0(X, \mathrm{gr}^0 \underline{\tilde{\mathrm{tr}}}'_{\tilde{\mathscr{F}}}(pu)) && \text{by (3.33)} \\
&= \mathrm{H}^0(X, \oplus_i \underline{\mathrm{tr}}'_{\mathrm{gr}^i \tilde{\mathscr{F}}}(\mathrm{gr}^i u)) && \text{by step 2 and 3.25} \\
&= \sum_i \mathrm{H}^0(X, \underline{\mathrm{tr}}'_{\mathrm{gr}^i \tilde{\mathscr{F}}}(\mathrm{gr}^i u)) \\
&= \sum_i \mathrm{tr}'_{\mathrm{gr}^i \tilde{\mathscr{F}}}(\mathrm{gr}^i u),
\end{aligned}$$

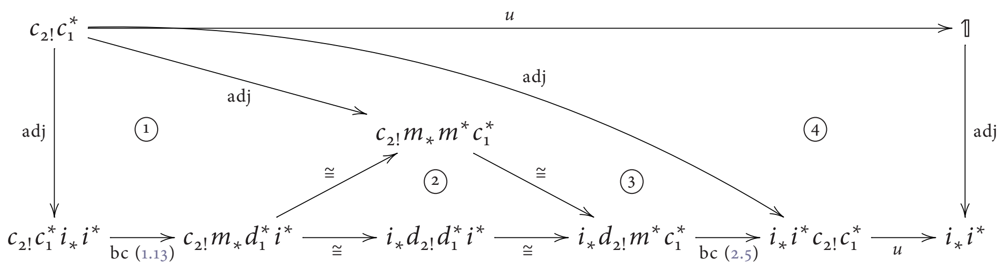
which is equal to the right hand side of (3.32). □

To apply the proposition to 3.23, let us go back to the situation at the beginning of this paragraph. Thus let $i : Z \hookrightarrow X$ be a closed subscheme, let $U = X \backslash Z$, $j : U \hookrightarrow X$. For any $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ we abbreviate $\mathscr{F}_Z := i_! i^* \mathscr{F}$ and $\mathscr{F}_U := j_! j^* \mathscr{F}$. Also, let $c : C \to X \times X$ be a correspondence such that $c|^Z$ exists. Notice first the following simple fact.

Lemma 3.32 *The following diagram commutes for any* $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$*:*

$$\begin{array}{ccc} c_{2!}c_1^*\mathscr{F}_U & \xrightarrow{[j^U]_!(u|^U)} & \mathscr{F}_U \\ \big\downarrow{\scriptstyle \mathrm{adj}} & & \big\downarrow{\scriptstyle \mathrm{adj}} \\ c_{2!}c_1^*\mathscr{F} & \xrightarrow{u} & \mathscr{F} \\ \big\downarrow{\scriptstyle \mathrm{adj}} & & \big\downarrow{\scriptstyle \mathrm{adj}} \\ c_{2!}c_1^*\mathscr{F}_Z & \xrightarrow[{[i^Z]_!(u|^Z)}]{} & \mathscr{F}_Z \end{array}$$

Proof We will show only that the lower square commutes because the argument for the upper case is very similar. Denote the inclusion $c_2^{-1}(Z)_{\mathrm{red}} \hookrightarrow C$ by m and set $d = c|^Z$. We use the definition of $[i^Z]_!(u|^Z)$ to decompose the diagram as follows (we abstain from writing $\mathscr{F}$):



The subdiagrams ② and ④ clearly commute, while ① and ③ commute by the definition of the base change morphisms. This is clear in the case of (1.13), and the case (2.5) is easily reduced to the former. □

Lemma 3.33 1. *The forgetful functor* $\omega : \mathfrak{D}^b_{ctf}\mathfrak{f}^{[0,1]}(X) \to \mathfrak{D}^b_{ctf}(X)$ *admits a natural section* $\mathscr{F} \mapsto \tilde{\mathscr{F}}$ *satisfying* $\mathrm{gr}^0\tilde{\mathscr{F}} = \mathscr{F}_Z$ *and* $\mathrm{gr}^1\tilde{\mathscr{F}} = \mathscr{F}_U$*.*

2. *The group morphism* $\omega : \mathrm{Hom}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}}, \tilde{\mathscr{F}}) \to \mathrm{Hom}(c_{2!}c_1^*\mathscr{F}, \mathscr{F})$ *admits a unique section* $u \mapsto \tilde{u}$ *and this section satisfies* $\mathrm{gr}^0\tilde{u} = [i^Z]_!(u|^Z)$ *and* $\mathrm{gr}^1\tilde{u} = [j^U]_!(u|^U)$*.*

Proof 1. Let us define this section on the level of complexes first. Denote by **A** the category of sheaves of Λ-modules for the étale topology on X. Then we define a functor $\mathfrak{C}^b(\mathbf{A}) \to \mathfrak{C}^b\mathfrak{f}^{[0,1]}(\mathbf{A})$ as follows. If $\mathscr{F}$ is an element of $\mathfrak{C}^b(\mathbf{A})$ then we cannot but set

$$F^i\tilde{\mathscr{F}} := \begin{cases} \mathscr{F} & : i \le 0 \\ \mathscr{F}_U & : i = 1 \\ 0 & : i > 1 \end{cases}$$

as the filtration on $\tilde{\mathscr{F}}$. Since a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ in $\mathfrak{C}^b(\mathbf{A})$ maps $j_!j^*\mathscr{F}$ into $j_!j^*\mathscr{G}$, we may (and have to) set $\tilde{\varphi} = \varphi$ to get the envisioned section on the level of complexes. Notice that since both functors $j_!$ and j^* are exact, if φ is an quasi-isomorphism then so are $F^0\tilde{\varphi} = \varphi$ and $F^1\tilde{\varphi} = j_!j^*\varphi$ hence the section descends to the derived categories as required.

2. We have to prove ω is an isomorphism. But the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}}) & \xrightarrow{\cong} & \mathrm{H}^0(\omega\tau^{\geq 0}\widetilde{\mathbf{R}\mathrm{Hom}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}})) \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \tau^{\geq 0}\to \mathbb{1}} \\
\mathrm{Hom}(c_{2!}c_1^*\mathscr{F},\mathscr{F}) & \xrightarrow[\cong]{} & \mathrm{H}^0(\omega\widetilde{\mathbf{R}\mathrm{Hom}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}}))
\end{array}$$

In view of the distinguished triangle $\tau^{\geq 0} \to \mathbb{1} \to \tau^{\leq -1} \to^+$ it thus suffices to show the vanishing of $\tau^{\leq -1}\widetilde{\mathbf{R}\mathrm{Hom}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}})$. Since $\widetilde{\mathbf{R}\mathrm{Hom}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}})$ belongs to $\mathfrak{D}^b_{ctf}\mathscr{F}^{[-1,1]}(X)$, we are reduced to showing the vanishing of

$$\begin{aligned}
\mathrm{gr}^{-1}\widetilde{\mathbf{R}\mathrm{Hom}}(\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\tilde{\mathscr{F}}) &\cong \oplus_i \mathbf{R}\mathrm{Hom}(\mathrm{gr}^{i+1}\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\mathrm{gr}^i\tilde{\mathscr{F}}) \\
&\cong \mathbf{R}\mathrm{Hom}(\mathrm{gr}^1\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}},\mathrm{gr}^0\tilde{\mathscr{F}}) \\
&\cong \mathbf{R}\mathrm{Hom}(c_{2!}c_1^*\mathscr{F}_U,\mathscr{F}_Z) \\
&\cong \mathbf{R}\mathrm{Hom}(i^*c_{2!}c_1^*\mathscr{F}_U,i^*\mathscr{F}).
\end{aligned}$$

Now, $c_2(c_1^{-1}(U))$ is contained in U (cf. 2.4.3) hence so is the support of $c_{2!}c_1^*\mathscr{F}_U$ implying that $i^*c_{2!}c_1^*\mathscr{F}_U = 0$.

Let $u \mapsto \tilde{u}$ be defined as the inverse map to ω and consider, for a fixed $u \in \mathrm{Hom}(c_{2!}c_1^*\mathscr{F},\mathscr{F})$, the following diagram:

$$\begin{array}{ccc}
\mathrm{gr}^1\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}} & \xrightarrow{\mathrm{gr}^1\tilde{u}} & \mathrm{gr}^1\tilde{\mathscr{F}} \\
\downarrow & & \downarrow \\
\omega\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}} & \xrightarrow{\omega\tilde{u}} & \omega\tilde{\mathscr{F}} \\
\downarrow & & \downarrow \\
\mathrm{gr}^0\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}} & \xrightarrow[\mathrm{gr}^0\tilde{u}]{} & \mathrm{gr}^0\tilde{\mathscr{F}}
\end{array}$$

where the vertical arrows are the canonical ones. Clearly, both squares commute. Now, it follows immediately from the definition of χ^i that the vertical maps in this diagram correspond to the adjunction maps $j_!j^! \to \mathbb{1}$ and $\mathbb{1} \to i_*i^*$, respectively, under the identification $\mathrm{gr}^1\tilde{c}_{2!}\tilde{c}_1^*\tilde{\mathscr{F}} \cong c_{2!}c_1^*\mathscr{F}_U$ and so on; i. e. we have a commutative diagram as in the previous Lemma (3.32) but where the top and bottom horizontal arrows correspond to $\mathrm{gr}^1\tilde{u}$ and $\mathrm{gr}^0\tilde{u}$, respectively. In view of the previous lemma, it thus suffices to prove that the top and bottom horizontal arrows of such a commutative diagram are unique. But we have already proved this. Indeed, the distinguished triangle $\mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Z \to^+$ gives rise to isomorphisms $\mathrm{Hom}(c_{2!}c_1^*\mathscr{F}_U,\mathscr{F}_U) \cong \mathrm{Hom}(c_{2!}c_1^*\mathscr{F}_U,\mathscr{F})$ and $\mathrm{Hom}(c_{2!}c_1^*\mathscr{F}_Z,\mathscr{F}_Z) \cong \mathrm{Hom}(c_{2!}c_1^*\mathscr{F},\mathscr{F}_Z)$ since $\mathbf{R}\underline{\mathrm{Hom}}(c_{2!}c_1^*\mathscr{F}_U,\mathscr{F}_Z) \cong 0$. □

Proof (3.23) Choose $\tilde{\mathscr{F}}$ and $\tilde{u}$ as in the previous lemma. Then:

$$\begin{aligned}
\mathrm{tr}(u) &= \mathrm{tr}(\omega\tilde{u}) \\
&= \mathrm{tr}_{\mathrm{gr}^0\tilde{\mathscr{F}}}(\mathrm{gr}^0\tilde{u}) + \mathrm{tr}_{\mathrm{gr}^1\tilde{\mathscr{F}}}(\mathrm{gr}^1\tilde{u}) && \text{by 3.31} \\
&= \mathrm{tr}_{\mathscr{F}_Z}([i^Z]_!(u|^Z)) + \mathrm{tr}_{\mathscr{F}_U}([j^U]_!(u|^U)) && \text{by 3.33.}
\end{aligned}$$
□

4 Main results

§1 Invariance

Let $c : C \to X \times X$ be a correspondence and let $Z \subset X$ be a closed subset. There are several ways to make precise the notion of Z being "invariant under c" three of which we will present below.

Definition 4.1 Let $c : C \to X \times X$ be a correspondence and let $Z \subset X$ be a closed subset. We say that Z is *c-invariant* if $c_1(c_2^{-1}(Z)) \subset_t Z$.

Here the subscript t is used to emphasize the fact that this inclusion should hold in the category of topological spaces (or sets). We will, in similar situations, use s as a subscript if the relation holds scheme-theoretically.

Remark 4.2 Let c and Z be as above but assume that Z is, in addition, given the structure of a closed subscheme of X. Notice that the following conditions are equivalent:

1. $c_1(c_2^{-1}(Z)) \subset_t Z$, i. e. Z is c-invariant;
2. $c_2^{-1}(Z) \subset_t c_1^{-1}(Z)$;
3. $c_2^{-1}(Z)_{\mathrm{red}} \subset_s c_1^{-1}(Z)$;
4. $\mathscr{I}_{c_1^{-1}(Z)} \subset \mathscr{I}_{c_2^{-1}(Z)_{\mathrm{red}}}$.

For our purposes the notion of invariance introduced above is too strong and we will want to have at our disposal two (weaker) notions of a more local character.

Definition 4.3 Let c and Z be as in the previous definition.

1. Z is said to be *locally c-invariant* if each $x \in Z$ possesses an open neighborhood $U \subset X$ such that $Z \cap U$ is $c|^U$-invariant.
2. Z is said to be *c-invariant in a neighborhood of fixed points* if there is an open neighborhood $W \subset C$ of $\mathrm{Fix}(c)$ such that Z is $c|_W$-invariant.

($c|^U$ and $c|_W$ were defined in 2.4.)

Remark 4.4 Spelling out the first condition in the definition we see that Z is locally c-invariant if and only if every $x \in Z$ possesses an open neighborhood U satisfying

$$\begin{aligned} (c|^U)_1((c|^u)_2^{-1}(Z \cap U)) &\subset_t Z \cap U, && \text{iff} \\ c_1(c_2^{-1}(Z \cap U)) \cap U &\subset_t Z \cap U, && \text{iff} \\ c_1(c_2^{-1}(Z \cap U)) &\subset_t Z \cup (X \backslash U). \end{aligned}$$

Notice also that every c-invariant subset is automatically locally c-invariant and c-invariant in a neighborhood of fixed points.

Example 4.5 Let $c : C \to X \times X$ be a correspondence and let $x \in X$ be a closed point such that $c_2^{-1}(x)$ is finite. Then $\{x\}$ is locally c-invariant. Indeed, the space $c_2^{-1}(x)$ is automatically discrete. Let $y \in c_2^{-1}(x)$ be any (necessarily closed) point and $z = c_1(y) \in X$. c_1 induces an inclusion of residue fields $k(z) \hookrightarrow k(y)$ which shows that $k(z)$ is contained in a finite extension of k hence that z is a closed point. This implies that $U = X \backslash (c_1(c_2^{-1}(x)) \backslash x)$ is an open neighborhood of x in X. Clearly, $\{x\}$ is $c|^U$-invariant.

Lemma 4.6 *Let $c : C \to X \times X$ be a correspondence and let $Z \subset X$ be a closed subset. There is a largest open subset $W \subset C$ such that Z is $c|_W$-invariant. Explicitly, $W = C \backslash \overline{c_2^{-1}(Z) \backslash c_1^{-1}(Z)}$, where $\overline{}$ denotes the Zariski-closure in C.*

Proof This is easy. □

Notation 4.7 We will denote this open subset by $W_c(Z)$.

Notation 4.8 Let $c : C \to X \times X$ be a correspondence, $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$ where $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and let $Z \subset X$ be a closed subset. In the case that Z is c-invariant we have defined in 2.13.3 the restriction $u|^Z = [i^Z]^* u \in \mathrm{Hom}_{c|^Z}(\mathscr{F}|_Z, \mathscr{F}|_Z)$. For general Z, let $W = W_c(Z) \subset C$. We then set $c\|^Z := (c|_W)|^Z$ and $u\|^Z := (u|_W)|^Z$.

Example 4.9 1. Let $c : C \to X \times X$ be a correspondence and assume that c_2 is quasi-finite. For each closed point $x \in X$, we have the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad c \quad} & X \times X \\ \uparrow{\scriptstyle j} & & \| \\ C\backslash(c_2^{-1}(x)\backslash c_1^{-1}(x)) & \xrightarrow[c|_{W_c(x)}]{} & X \times X \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i_x \times i_x} \\ (c_2^{-1}(x) \cap c_1^{-1}(x))_{\mathrm{red}} & \xrightarrow[c\|^x]{} & \{x\} \times \{x\} \end{array}$$

with $\mathrm{Fix}(c\|^x) = (c_1^{-1}(x) \cap c_2^{-1}(x))_{\mathrm{red}}$ a finite scheme.

Now, let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$, $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$, and $y \in \mathrm{Fix}(c\|^x)$. Then the restriction of $u\|^x$ to $\{y\} \to \{x\} \times \{x\}$ defines a map

$$u\|^x|_y : \mathscr{F}_x \longrightarrow \mathscr{F}_x$$

and we claim that this map can be identified with the restriction of the stalk map

$$u_x : (c_{2!} c_1^* \mathscr{F})_x \cong \oplus_{c_2(z)=x} (c_1^* \mathscr{F})_z \to \mathscr{F}_x$$

to $(c_1^* \mathscr{F})_y = \mathscr{F}_x$. The latter map will be denoted u_y.

To see the identification, set $e = c|_{W_c(x)}$ hence $e_2^{-1}(x) = c_2^{-1}(x) \cap c_1^{-1}(x)$. Notice that the inclusion $\oplus_{z \in c_2^{-1}(x) \cap c_1^{-1}(x)} (c_1^* \mathscr{F})_z \hookrightarrow \oplus_{z \in c_2^{-1}(x)} (c_1^* \mathscr{F})_z$ is induced by the morphism

$$\varphi : e_{2!} e_1^* \cong c_{2!} j_! j^* c_1^* \xrightarrow{\mathrm{adj}} c_{2!} c_1^*,$$

while one easily sees that $u|_{W_c(x)} = u \varphi_{\mathscr{F}}$. This means that $(u|_{W_c(x)})_x$ factors through u_x and we have a canonical identification of $(u|_{W_c(x)})_y$ with u_y:

$$\begin{array}{ccccc} (c_1^* \mathscr{F})_y & \hookrightarrow & \oplus_{z \in c_2^{-1}(x)} (c_1^* \mathscr{F})_z & \xrightarrow{\quad u_x \quad} & \mathscr{F}_x \\ \| & & \uparrow{\scriptstyle (\varphi_{\mathscr{F}})_x} & & \| \\ (c_1^* \mathscr{F})_y & \hookrightarrow & \oplus_{z \in c_2^{-1}(x) \cap c_1^{-1}(x)} (c_1^* \mathscr{F})_z & \xrightarrow[(u|_{W_c(x)})_x]{} & \mathscr{F}_x \end{array}$$

Hence we may from now on assume x to be c-invariant.

Set $d = c||^x = [i^x]^*c$. The argument is similar as above: The map $\oplus_{z\in d_2^{-1}(x)}(d_1^*\mathscr{F})_z \to \oplus_{z\in c_2^{-1}(x)}(c_1^*\mathscr{F})_z$ is induced by a morphism $\varphi' : d_{2!}d_1^* \to c_{2!}c_1^*$ such that $u||^x = u\varphi'_{\mathscr{F}}$ hence there is an identification of $(u||^x)_y$ with u_y, i. e. we may assume $X = \{x\}$.

Exactly the same argument as before (replacing j by the inclusion $y \hookrightarrow C$ and $u|_{W_c(x)}$ by $u|_y$) gives an identification of $(u|_y)_y$ with u_y hence we may assume $C = \{y\}$. But in this case the claim is obvious.

2. As a slight generalization, suppose c_2 is quasi-finite only on some open neighborhood $C' \subset C$ of $\mathrm{Fix}(c||^x)$. There is a natural way to define u_y for $y \in \mathrm{Fix}(c||^x)$ analogously as above, namely by $u_y := (u|_{C'})_y$. We also have an identification

$$\begin{aligned} u|_{C'}||^x &= u|_{C'}|_{W_{c|_{C'}}(x)}|^x \\ &= u|_{W_{c|_{C'}}(x)}|^x && \text{by 2.12} \\ &= u|_{W_c(x)}|^x|_{W_c(x)\cap C'\cap c_2^{-1}(x)_{\mathrm{red}}} && \text{by 2.14} \\ &= u||^x && \text{since } W_c(x)\cap c_2^{-1}(x)_{\mathrm{red}} = W_c(x)\cap C'\cap c_2^{-1}(x)_{\mathrm{red}}. \end{aligned}$$

The above argument thus shows again that u_y can be identified with $u||^x|_y$.

Moreover, if $C'' \subset C'$ is another open neighborhood of $\mathrm{Fix}(c||^x)$ then $(u|_{C''})_y$ is canonically identified with $(u|_{C'})_y$. Hence it makes sense to set $u_y := (u|_{C'})_y$ for *any* open neighborhood C' of $\mathrm{Fix}(c||^x)$ on which c_2 is quasi-finite, provided that such a neighborhood exists.

Our loose way of speaking about "identifying" certain morphisms should cause no worries since the only thing we're interested in is their trace:

Definition 4.10 In the notation of the previous example, $u_y : \mathscr{F}_x \to \mathscr{F}_x$ may be assigned a trace $\mathrm{Tr}(u_y)$ (as in 3.3), called the *naive local term of u at y*.

Remark 4.11 Our argument above shows that $\mathrm{lt}_y(u||^x) = \mathrm{Tr}(u_y)$, i. e. "the naive local term equals the real local term". Indeed,

$$\begin{aligned} \mathrm{lt}_y(u||^x) &= \mathrm{res}_y(\mathrm{tr}(u||^x)) && \text{since } \pi_y = \mathbb{1} \\ &= \mathrm{tr}([j_y]^*u||^x) && \text{by 3.22} \\ &= \mathrm{tr}(u_y) && \text{by what we just showed} \\ &= \mathrm{Tr}(u_y) && \text{by 3.2.} \end{aligned}$$

In the generalization discussed in the previous example, i. e. c_2 quasi-finite on an open neighborhood $C' \subset C$ of $\mathrm{Fix}(c||^x)$, we thus get the identity $\mathrm{lt}_y(u||^x) = \mathrm{lt}_y(u|_{C'}||^x) = \mathrm{Tr}(u_y)$.

Let us now give alternative characterizations of local invariance and invariance in a neighborhood of fixed points which will be useful later on.

Lemma 4.12 *Let $c : C \to X \times X$ be a correspondence and let $Z \subset X$ be a closed subset. Set $B = c_2^{-1}(Z)\backslash c_1^{-1}(Z)$.*

1. *Z is locally c-invariant if and only if for every irreducible component S of B,*

$$\overline{c_1(S)} \cap \overline{c_2(S)} = \varnothing.$$

2. *Z is c-invariant in a neighborhood of fixed points if and only if $\overline{B} \cap \mathrm{Fix}(c) = \varnothing$.*

3. *If Z is locally c-invariant, then it is also c-invariant in a neighborhood of fixed points.*

Proof 1. For an open subset $U \subset X$, the last condition in 4.4 holds if and only if

$$B = c_2^{-1}(Z) \backslash c_1^{-1}(Z) \subset_t c_1^{-1}(X \backslash U) \cup c_2^{-1}(X \backslash U),$$

which in turn holds if and only if for every irreducible component S of B, $c_1(S) \subset_t X \backslash U$ or $c_2(S) \subset_t X \backslash U$. Thus Z is locally c-invariant if and only if for every $x \in Z$ and every S as above, $x \notin \overline{c_1(S)}$ or $x \notin \overline{c_2(S)}$, i. e. if and only if $Z \cap \overline{c_1(S)} \cap \overline{c_2(S)} = \varnothing$. But since $c_2(S) \subset_t Z$, $Z \cap \overline{c_1(S)} \cap \overline{c_2(S)} =_t \overline{c_1(S)} \cap \overline{c_2(S)}$.

2. This is clear by 4.6.

3. Assume Z locally c-invariant and let $x \in \overline{B}$. By part 2, we have to show $x \notin \mathrm{Fix}(c)$. Let S be an irreducible component of B such that $x \in \overline{S}$. Then, by part 1,

$$c_1(x) \in c_1(\overline{S}) \subset_t \overline{c_1(S)} \subset_t X \backslash \overline{c_2(S)} \subset_t X \backslash c_2(\overline{S}) \subset_t X \backslash c_2(x)$$

hence $x \notin \mathrm{Fix}(c)$. □

We end this paragraph with a result concerning the behavior of local invariance with respect to compactification.

Lemma 4.13 *Let $c : C \to X \times X$ be a correspondence and $U \subset X$ an open subset such that $c_1^{-1}(U)$ is dense in C, $c_1|_{c_1^{-1}(U)}$ is proper, and $X \backslash U$ is locally c-invariant.*

1. *Every compactification $\overline{c} : \overline{C} \to \overline{X} \times \overline{X}$ of c satisfies $\overline{c}_1^{-1}(U) = c_1^{-1}(U)$.*

2. *There exists a compactification $\overline{c} : \overline{C} \to \overline{X} \times \overline{X}$ of c such that $\overline{X} \backslash U$ is locally $\overline{c}$-invariant.*

Proof 1. Notice that $c_1^{-1}(U)$ is a dense open subset of $\overline{c}_1^{-1}(U)$. Moreover, $\overline{c}_1|_{c_1^{-1}(U)} = c_1|_{c_1^{-1}(U)}$ is proper and $\overline{c}_1|_{\overline{c}_1^{-1}(U)}$ is separated hence the inclusion $c_1^{-1}(U) \hookrightarrow \overline{c}_1^{-1}(U)$ is proper thus surjective.

2. Set $Z := X \backslash U$. We claim there exists a compactification $\overline{X}$ of X such that for each irreducible component S of $c_2^{-1}(Z) \backslash c_1^{-1}(Z)$, the closures of $c_1(S)$ and $c_2(S)$ in $\overline{X}$ do not intersect.

   Let $S_1, \ldots, S_n$ be the irreducible components of $c_2^{-1}(Z) \backslash c_1^{-1}(Z)$. We deduce an n-tuple of pairs $(C_{1,i}, C_{2,i})$ where $C_{j,i}$ is the scheme-theoretic closure of $c_j(S_i)$ in X. Each pair satisfies

$$C_{1,i} \cap C_{2,i} = \varnothing \tag{4.1}$$

   by 4.12.1. We will prove now more generally that given such data, i. e. a scheme X and an n-tuple of pairs of closed subschemes $(C_{1,i}, C_{2,i})$ satisfying (4.1), there exists a compactification $\overline{X}$ of X such that the closures of the $C_{j,i}$ in $\overline{X}$ still satisfy (4.1).

   We prove this by induction on n. If $n = 0$, we may choose $\overline{X}$ to be any compactification of X. So let $n \geq 1$ and, using the induction hypothesis, choose a compactification $\overline{X}'$ of X such that the closures of the first $n-1$ pairs satisfy (4.1). Possibly, we have $\overline{C_{1,n}}' \cap \overline{C_{2,n}}' \neq \varnothing$ ($\overline{(-)}'$ being the closure operation in $\overline{X}'$). In this case, we let $\overline{X}$ be the blow-up of $\overline{X}'$ with center

$$\mathscr{J} = \mathscr{I}_{\overline{C_{1,n}}'} + \mathscr{I}_{\overline{C_{2,n}}'}.$$

This is still a compactification of X. The closures of $C_{1,n}$ and $C_{2,n}$ in $\overline{X}$ do not intersect as they are contained in the strict transforms of $\overline{C_{1,n}}'$ and $\overline{C_{1,n}}'$, respectively, which do not intersect. By the lemma below (applied $n-1$ times), the closures of the first $n-1$ pairs still satisfy (4.1), which concludes the induction step and hence the proof of the more general claim above.

Now, choose a compactification $\overline{X}$ as in the claim above and extend it to a compactification $\overline{c}:\overline{C}\to\overline{X}\times\overline{X}$ (2.5.1). By part 1, $\overline{c}_1^{-1}(U)=c_1^{-1}(U)$, and this implies

$$\begin{aligned}\overline{c}_2^{-1}(\overline{X}\backslash U)\backslash\overline{c}_1^{-1}(\overline{X}\backslash U) &= \overline{c}_2^{-1}(\overline{X}\backslash U)\cap\overline{c}_1^{-1}(U)\\ &= \overline{c}_2^{-1}(\overline{X}\backslash U)\cap c_1^{-1}(U)\\ &= c_2^{-1}(X\backslash U)\backslash c_1^{-1}(X\backslash U)\\ &= c_2^{-1}(Z)\backslash c_1^{-1}(Z).\end{aligned}$$

By our choice of $\overline{X}$, for every irreducible component S of $\overline{c}_2^{-1}(\overline{X}\backslash U)\backslash\overline{c}_1^{-1}(\overline{X}\backslash U)$, the closures of $\overline{c}_1(S)=c_1(S)$ and $\overline{c}_2(S)=c_2(S)$ in $\overline{X}$ do not intersect. 4.12.1 now tells us that $\overline{X}\backslash U$ is locally $\overline{c}$-invariant. □

Lemma 4.14 *Let X be a scheme, let Y, Z be two closed disjoint subschemes, and let $\mathscr{J}\subset\mathscr{O}_X$ be a sheaf of ideals. Let $\overline{X}$ be the blow-up of X with center $\mathscr{J}$ and let $\overline{Y}$ and $\overline{Z}$ be the strict transforms of Y and Z, respectively. Then $\overline{Y}\cap\overline{Z}=\varnothing$ in $\overline{X}$.*

Proof Set $\mathscr{S}=\oplus_{n\geq 0}\mathscr{J}^n$ so that $\overline{X}=\mathbf{Proj}(\mathscr{S})$. The subscheme $\overline{Y}\subset\overline{X}$ is defined by the sheaf of ideals associated to the graded $\mathscr{S}$-module

$$\mathscr{K}_Y:=\ker\Big(\oplus_{n\geq 0}\mathscr{J}^n\longrightarrow\oplus_{n\geq 0}(\mathscr{I}_Y+\mathscr{J}^n)/\mathscr{I}_Y\Big)=\oplus_{n\geq 0}(\mathscr{J}^n\cap\mathscr{I}_Y)$$

(see [11, 3.6.2 (i)]). But for each $n\geq 0$,

$$\mathscr{J}^n\cap\mathscr{I}_Y+\mathscr{J}^n\cap\mathscr{I}_Z=\mathscr{J}^n$$

by assumption, i. e. $\widetilde{\mathscr{K}_Y}+\widetilde{\mathscr{K}_Z}=\widetilde{\mathscr{S}}=\mathscr{O}_{\overline{X}}$. □

§2 Contracting correspondences

Definition 4.15 Let $c:C\to X\times X$ be a correspondence and let $Z\subset X$ be a closed subscheme.

1. *c stabilizes Z* if $c_1(c_2^{-1}(Z))\subset_s Z$. (Here the left hand side denotes the scheme theoretic image of $c_1|_{c_2^{-1}(Z)}$.)
2. c is *contracting near Z* if c stabilizes Z and if there exists $k_0\in\mathbb{N}$ such that
$$(c_1^*(\mathscr{I}_Z)\cdot\mathscr{O}_C)^{k_0}\subset(c_2^*(\mathscr{I}_Z)\cdot\mathscr{O}_C)^{k_0+1}.$$
3. c is *contracting near Z in a neighborhood of fixed points*, if there exists an open neighborhood $W\subset C$ of $\mathrm{Fix}(c)$ such that $c|_W$ is contracting near Z.

Remark 4.16 Let c and Z be as in the definition. Then the following conditions are equivalent:

1. $c_1(c_2^{-1}(Z))\subset_s Z$, i. e. c stabilizes Z;

2. $c_2^{-1}(Z) \subset_s c_1^{-1}(Z)$;

3. $c^{-1}(Z \times Z) =_s c_2^{-1}(Z)$;

4. $c_1^* \mathscr{I}_Z \cdot \mathscr{O}_C \subset c_2^* \mathscr{I}_Z \cdot \mathscr{O}_C$.

In particular, comparing condition 2 here with condition 3 of 4.2, we see immediately that if c stabilizes Z then Z is automatically c-invariant. Hence if c is contracting near Z in a neighborhood of fixed points then Z is c-invariant in a neighborhood of fixed points.

Proof The equivalence of 2 and 4 is obvious (cf. 1.(b)). If 2 holds then $c_1(c_2^{-1}(Z)) \subset_s c_1(c_1^{-1}(Z)) \subset_s Z$ and, conversely, if c stabilizes Z then $c_1|_{c_2^{-1}(Z)}$ factors through Z inducing a morphism $c_2^{-1}(Z) \to C \times_X Z = c_1^{-1}(Z)$, thus the closed immersion $c_2^{-1}(Z) \hookrightarrow C$ factors through $c_1^{-1}(Z)$. Finally, the equivalence of 2 and 3 follows from the fact that $c^{-1}(Z \times Z) = c_1^{-1}(Z) \times_C c_2^{-1}(Z)$.

Lemma 4.17 *Let $c : C \to X \times X$ be a correspondence and $Z \subset X$ a closed subscheme. c is contracting near Z if and only if c stabilizes Z and the image of $(\tilde{c}_Z)_{1s} : N_{c^{-1}(Z\times Z)}(C) \to N_Z(X)$ is contained set-theoretically in the zero section $Z \subset N_Z(X)$.*

Proof Consider the following composition of morphisms in **ahc** (2.29):

$$c_1 : (C, c^{-1}(Z \times Z)) \xrightarrow{c} (X \times X, Z \times Z) \xrightarrow{p_1} (X, Z).$$

By functoriality (2.30.1), $(\tilde{c}_Z)_1 = \tilde{p}_1 \circ \tilde{c}_Z = \widetilde{p_1 \circ c} = \tilde{c}_1$. By 2.30.2, the image of $\tilde{c}_{1s}$ is contained in the zero section if and only if there exists a $k_0 \in \mathbb{N}$ such that the following inclusion holds:

$$(c_1^*(\mathscr{I}_Z) \cdot \mathscr{O}_C)^{k_0} = c_1^*(\mathscr{I}_Z^{k_0}) \cdot \mathscr{O}_C \subset \mathscr{I}^{k_0+1}_{c^{-1}(Z\times Z)} = \mathscr{I}^{k_0+1}_{c_2^{-1}(Z)} = (c_2^*(\mathscr{I}_Z) \cdot \mathscr{O}_C)^{k_0+1}$$

In the penultimate equality we used that c stabilizes Z in any case (and 4.16). □

The reason for introducing the notion of contraction is the following result.

Theorem 4.18 *Let $c : C \to X \times X$ be a correspondence contracting near a closed subscheme $Z \subset X$ in a neighborhood of fixed points, let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ and let $u \in \mathrm{Hom}_c(\mathscr{F}, \mathscr{F})$ be a cohomological correspondence. Moreover, let β be an open connected subset of* $\mathrm{Fix}(c)$ *such that $c'(\beta) \cap Z \neq \varnothing$. Then:*

1. *$\beta \subset_t c'^{-1}(Z)$, i. e. β is an open connected subset of* $\mathrm{Fix}(c\|^Z)$.

2. $\mathrm{tr}_\beta(u) = \mathrm{tr}_\beta(u\|^Z)$. *In particular, if β is proper then* $\mathrm{lt}_\beta(u) = \mathrm{lt}_\beta(u\|^Z)$.

Proof 1. Let us first prove that the second statement in 1 follows from the first. For this notice that since Z is c-invariant in a neighborhood of fixed points (4.16), $\mathrm{Fix}(c) \subset_s W_c(Z)$ hence $c' = (c|_{W_c(Z)})'$. This implies $c'^{-1}(Z) \subset_t (c|_{W_c(Z)})_2^{-1}(Z)_{\mathrm{red}}$ and finally

$$\begin{aligned} \mathrm{Fix}(c|_{W_c(Z)}|^Z) &= (c|_{W_c(Z)})_2^{-1}(Z)_{\mathrm{red}} \times_{Z\times Z} Z \\ &= (c|_{W_c(Z)})_2^{-1}(Z)_{\mathrm{red}} \times_{X\times X} Z \\ &= (c|_{W_c(Z)})_2^{-1}(Z)_{\mathrm{red}} \times_C c'^{-1}(Z) \\ &=_t c'^{-1}(Z). \end{aligned}$$

Thus it suffices to prove the first statement.

Let $W \subset C$ be an open neighborhood of $\mathrm{Fix}(c)$ such that $c|_W$ is contracting near Z. Then, as before, $\mathrm{Fix}(c) = \mathrm{Fix}(c|_W)$ and $c' = (c|_W)'$ so we may replace c by $c|_W$, assuming from now on that c is contracting near Z. Also, replacing C by the open subset $C \backslash (\mathrm{Fix}(c) \backslash \beta)$ does not affect $c'|_\beta$ hence we may assume that $\mathrm{Fix}(c) = \beta$.

Since c is contracting near Z the previous lemma tells us that the image of the morphism $(\tilde{c}_Z)_{1s} : N_{c^{-1}(Z \times Z)}(C) \to N_Z(X)$ is contained set-theoretically in the zero section $Z \subset N_Z(X)$. Moreover, by 2.34, there is a commutative triangle

$$\begin{array}{ccc} \tilde{\beta}_{c'^{-1}(Z)} & & \\ \downarrow & \overset{\widetilde{c'}}{\searrow} & \\ \mathrm{Fix}(\tilde{c}_Z) & \xrightarrow[(\tilde{c}_Z)']{} & \tilde{X}_Z \end{array}$$

which shows, by passing to the fiber over s, that

$$\mathrm{Im}[\widetilde{c'}_s] \subset_t \mathrm{Im}[(\tilde{c}_Z)'_s] \subset_t \mathrm{Im}[(\tilde{c}_Z)_{1s}] \subset_t Z.$$

Now, we may use 2.30.2 to deduce the existence of $k_0 \in \mathbb{N}$ such that $(c'^*(\mathscr{I}_Z) \cdot \mathscr{O}_\beta)^{k_0} \subset (c'^*(\mathscr{I}_Z) \cdot \mathscr{O}_\beta)^{k_0+1}$, i. e. $\mathscr{I}^{k_0}_{c'^{-1}(Z)} = \mathscr{I}^{k_0+1}_{c'^{-1}(Z)}$. We conclude with the following Lemma 4.19 that $\mathscr{I}^{k_0}_{c'^{-1}(Z)} = 0$ hence $\beta =_t c'^{-1}(Z)$.

2. It suffices to prove the first statement. We will proceed in several steps.

Step 1 As in part 1, we may choose an open neighborhood $W \subset C$ of $\mathrm{Fix}(c)$ such that $c|_W$ is contracting near Z, and again we have $\mathrm{Fix}(c) = \mathrm{Fix}(c|_W)$. Suppose we can prove $\mathrm{tr}_\beta(u|_W) = \mathrm{tr}_\beta((u|_W)|^Z)$. Then we also have

$$\begin{aligned} \mathrm{tr}_\beta(u) &= \mathrm{tr}_\beta(u|_W) && \text{by 3.22} \\ &= \mathrm{tr}_\beta(u|_W|^Z) && \text{by hypothesis} \\ &= \mathrm{tr}_\beta(u|_{W_c(Z)}|^Z|_{W \cap c_2^{-1}(Z)_{\mathrm{red}}}) && \text{by 2.14} \\ &= \mathrm{tr}_\beta(u||^Z) && \text{by 3.22.} \end{aligned}$$

Thus we may from now on assume that c is contracting near Z.

Step 2 Let W be the open set $C \backslash (\mathrm{Fix}(c) \backslash \beta)$ and suppose that we can prove $\mathrm{tr}_{c|_W}(u|_W) = \mathrm{tr}_{c|_W|^Z}(u|_W|^Z)$. Then we also have

$$\begin{aligned} \mathrm{tr}_\beta(u) &= \mathrm{tr}_{c|_W}(u|_W) && \text{by 3.22} \\ &= \mathrm{tr}_{c|_W|^Z}(u|_W|^Z) && \text{by hypothesis} \\ &= \mathrm{tr}_{c|^Z|_{W \cap c_2^{-1}(Z)_{\mathrm{red}}}}(u|^Z|_{W \cap c_2^{-1}(Z)_{\mathrm{red}}}) && \text{by 2.14} \\ &= \mathrm{tr}_\beta(u|^Z) && \text{by 3.22 and part 1.} \end{aligned}$$

Thus we may from now on assume $\beta = \mathrm{Fix}(c)$.

Step 3 Set $U = X \backslash Z$. We may apply 3.23 to get the identity

$$\mathrm{tr}_{\mathscr{F}}(u) = \mathrm{tr}_{\mathscr{F}_Z}([i^Z]_!(u|^Z)) + \mathrm{tr}_{\mathscr{F}_U}([j^U]_!(u|^U)).$$

By 3.19, the first summand on the right is equal to $\int_{(i^Z)'} \mathrm{tr}_{c|^Z}(u|^Z)$. But since $\mathrm{Fix}(c) =_t \mathrm{Fix}(c|^Z)$ by part 1, the map $\int_{(i^Z)'} : \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \to \mathrm{H}^0(\mathrm{Fix}(c|^Z), K_{\mathrm{Fix}(c|^Z)})$ is the identity map. Hence the first summand is equal to $\mathrm{tr}_{c|^Z}(u|^Z)$. It suffices thus to show the vanishing of the second summand.

Step 4 By the previous step it suffices to show $\mathrm{tr}_c(u) = 0$ if $\mathscr{F}|_Z = 0$. Now, by 3.21, the following identity holds:

$$\mathrm{sp}_{\mathrm{Fix}(\tilde{c}_Z)}(\mathrm{tr}_c(u)) = \mathrm{tr}_{(\tilde{c}_Z)_s}(\mathrm{sp}_{\tilde{c}_Z}(u)). \tag{4.2}$$

As in part 1 (by 4.17), the image of $(\tilde{c}_Z)_{1s}$ is contained set-theoretically in $Z \subset N_Z(X)$. On the other hand, $\mathrm{sp}_{\tilde{X}_Z}(\mathscr{F})|_Z \cong \mathscr{F}|_Z = 0$ by 2.35, hence $(\tilde{c}_Z)_{1s}^*\mathrm{sp}_{\tilde{X}_Z}(\mathscr{F}) = 0$ as well. Looking at the definition of $\mathrm{sp}_{\tilde{c}_Z}$ (2.20), we see that this implies $\mathrm{sp}_{\tilde{c}_Z}(u) = 0$ hence the right hand side of the above identity (4.2) vanishes.

Step 5 For the proof of the theorem it will suffice to show that the map

$$\mathrm{sp}_{\mathrm{Fix}(\tilde{c}_Z)} : \mathrm{H}^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \to \mathrm{H}^0(\mathrm{Fix}((\tilde{c}_Z)_s), K_{\mathrm{Fix}((\tilde{c}_Z)_s)}) \tag{4.3}$$

in (4.2) is an isomorphism. By 2.24, this is true for $\mathrm{sp}_{(\mathrm{Fix}(c)_{\mathrm{red}})_R}$, and it is easy to see that the same must hold for (4.3) provided that

$$\mathrm{Fix}(\tilde{c}_Z)_{\mathrm{red}} \cong (\mathrm{Fix}(c)_{\mathrm{red}})_R \quad (\cong (\mathrm{Fix}(c)_R)_{\mathrm{red}}) \tag{4.4}$$

as schemes over $\mathrm{Fix}(c)_R$ (the two maps in question are related to each other by counits of adjunctions, $p_!p^! \to \mathbb{1}$, associated to closed immersions onto, p, i. e. by isomorphisms). In order to prove (4.4), consider the following canonical isomorphism over $\mathrm{Fix}(c)_R$:

$$\begin{aligned} (\mathrm{Fix}(c)_{\mathrm{red}})_R &= (\widetilde{\mathrm{Fix}(c)}_{\mathrm{red}})_{\mathrm{Fix}(c)_{\mathrm{red}}} \\ &= (\widetilde{\mathrm{Fix}(c)}_{\mathrm{red}})_{c'^{-1}(Z)_{\mathrm{red}}} && \text{by part 1} \\ &\cong (\widetilde{\mathrm{Fix}(c)}_{c'^{-1}(Z)})_{\mathrm{red}} && \text{by 2.28.5.} \end{aligned}$$

We deduce from 2.34 that $(\mathrm{Fix}(c)_R)_{\mathrm{red}}$ is a closed subscheme of $\mathrm{Fix}(\tilde{c}_Z)_{\mathrm{red}}$, the embedding being a morphism over $\mathrm{Fix}(c)_R$. Passing to the generic fiber of the cartesian square

$$\begin{array}{ccc} \mathrm{Fix}(\tilde{c}_Z) & \longrightarrow & \tilde{X}_Z \\ \downarrow & & \downarrow \Delta \\ \tilde{C}_{c^{-1}(Z\times Z)} & \xrightarrow[\tilde{c}_Z]{} & \tilde{X}_Z \times_R \tilde{X}_Z \end{array}$$

we see that $\mathrm{Fix}(\tilde{c}_Z)_\eta$ is isomorphic to $\mathrm{Fix}(c)_\eta$ hence the generic fibers of the two schemes $(\mathrm{Fix}(c)_R)_{\mathrm{red}}$ and $\mathrm{Fix}(\tilde{c}_Z)_{\mathrm{red}}$ are the same. It remains to prove (4.4) over the special fiber s.

Step 6 There is a chain of closed subsets

$$\mathrm{Fix}(c) = (\mathrm{Fix}(c)_R)_s \subset_t \mathrm{Fix}(\tilde{c}_Z)_s = \mathrm{Fix}((\tilde{c}_Z)_s) \subset_t N_{c^{-1}(Z\times Z)}(C),$$

and by the last step it suffices to prove that the first inclusion is in fact a (set-theoretical) equality.

Since the image of $\mathrm{Fix}((\tilde{c}_Z)_s)$ under $(\tilde{c}_Z)_{1s}$ is set-theoretically contained in Z, the same is true of $(\tilde{c}_Z)_{2s}$. On the other hand, 2.31.1 tells us that $(\tilde{c}_Z)_{2s}^{-1}(Z) = c_2^{-1}(Z) \subset N_{c^{-1}(Z\times Z)}(C)$.

Therefore we have the inclusion

$$\mathrm{Fix}((\tilde{c}_Z)_s) \subset c_2^{-1}(Z).$$

But by 2.30.3, $(\tilde{c}_Z)_s|_{c_2^{-1}(Z)}$ is equal to $c|^Z$ (at least as maps between topological spaces) from which we conclude that $\mathrm{Fix}((\tilde{c}_Z)_s) \subset_t \mathrm{Fix}(c|^Z) \subset_t \mathrm{Fix}(c)$. □

Lemma 4.19 *Let X be a connected noetherian scheme, let $k_0 \in \mathbb{N}$ and let $\mathscr{F}$ be a sheaf of ideals satisfying: 1. $\mathscr{F} \neq \mathscr{O}_X$; 2. $\mathscr{F}^{k_0} = \mathscr{F}^{k_0+1}$. Then $\mathscr{F}^{k_0} = 0$.*

Proof Set $\mathscr{G} = \mathscr{F}^{k_0}$. We have the equality

$$\{x \in X \mid \mathscr{G}_x = 0\} = \{x \in X \mid \mathscr{O}_{X,x}/\mathscr{G}_x \neq 0\}$$

since, if $\mathscr{G}_x \neq \mathscr{O}_{X,x}$ then we also have $\mathscr{F}_x \neq \mathscr{O}_{X,x}$ and $\mathscr{F}_x\mathscr{G}_x = \mathscr{G}_x$ hence Nakayama's lemma (which may be applied because X is noetherian) implies $\mathscr{G}_x = 0$. Now the first space above is open, the second is closed and non-empty by assumption hence $\mathscr{G}_x = 0$ everywhere. □

§3 Correspondences over finite fields

We now specialize to the case where $k = \mathbb{F}$ is an algebraic closure of a finite field $\mathbb{F}_q$. If $\overline{X}$ is a scheme over $\mathbb{F}$ we say that it is *defined over* $\mathbb{F}_q$ if there exists a scheme X over $\mathbb{F}_q$ such that $\overline{X} = X \times_{\mathbb{F}_q} \mathbb{F}$. Similarly, a morphism of schemes over $\mathbb{F}$, $\overline{f} : \overline{X} \to \overline{Y}$, is said to be *defined over* $\mathbb{F}_q$ if there exists a morphism of schemes over $\mathbb{F}_q$, $f : X \to Y$, such that $\overline{f} = f \times_{\mathbb{F}_q} \mathbb{F}$. We first recall the definition of several Frobenius morphisms.

Let X be a scheme over $\mathbb{F}_q$. The *absolute Frobenius* of X over $\mathbb{F}_q$, $\mathrm{Fr}_X : X \to X$, is defined to be the identity on the topological space X and locally, for $\mathrm{Spec}(A) \subset X$ an open affine, by the ring morphism $a \mapsto a^q$, $a \in A$, on the sheaves. Let $\overline{X} = X \times_{\mathbb{F}_q} \mathbb{F}$ be the extension of scalars. The *geometric Frobenius* of $\overline{X}$ over $\mathbb{F}_q$, $\mathrm{Fr}_{X,g} : \overline{X} \to \overline{X}$, is the morphism $\mathrm{Fr}_X \times_{\mathbb{F}_q} \mathbb{1}_{\mathbb{F}}$. The *arithmetic Frobenius* of $\overline{X}$ over $\mathbb{F}_q$ on the other hand is $\mathrm{Fr}_{X,a} = \mathbb{1}_X \times_{\mathbb{F}_q} \mathrm{Fr}_{\mathbb{F}} : \overline{X} \to \overline{X}$. The relationship between these various Frobenius morphisms is

$$\mathrm{Fr}_{X,a} \circ \mathrm{Fr}_{X,g} = \mathrm{Fr}_{X,g} \circ \mathrm{Fr}_{X,a} = \mathrm{Fr}_{\overline{X}}. \tag{4.5}$$

Notice also that $\mathrm{Fr}_{\mathbb{F}}$ is an automorphism of $\mathbb{F}$ hence $\mathrm{Fr}_{X,a}$ is an automorphism of $\overline{X}$.

Definition 4.20 Let $c : C \to X_1 \times X_2$ be a correspondence defined over $\mathbb{F}_q$ and let $n \in \mathbb{N}$. The correspondence

$$c^{(n)} := (c_1^{(n)}, c_2) := (\mathrm{Fr}^n_{X_1,g} \circ c_1, c_2) : C \to X_1 \times X_2$$

is called the n^{th} *twist of c by the Frobenius*.

Remark 4.21 It is clear that the association $X \mapsto \mathrm{Fr}_X$ defines a natural endomorphism of the identity functor on the $\mathbf{Sch}/\mathbb{F}_q$. This simply means that for a morphism $f : X \to Y$ of schemes over $\mathbb{F}_q$ we have $\mathrm{Fr}_Y \circ f = f \circ \mathrm{Fr}_X$. From this it follows that

$$\begin{aligned}\mathrm{Fr}_{Y,g} \circ \overline{f} &= (\mathrm{Fr}_Y \times \mathbb{1}_{\mathbb{F}}) \circ (f \times \mathbb{1}_{\mathbb{F}}) \\ &= (\mathrm{Fr}_Y \circ f) \times \mathbb{1}_{\mathbb{F}} \\ &= (f \circ \mathrm{Fr}_X) \times \mathbb{1}_{\mathbb{F}} \\ &= \overline{f} \circ \mathrm{Fr}_{X,g},\end{aligned}$$

i. e. the association $X \mapsto \mathrm{Fr}_{X,g}$ defines a natural endomorphism of the functor $-\times_{\mathbb{F}_q}\mathbb{F}$. In particular, we see that the n^{th} twist of a correspondence $c : C \to X_1 \times X_2$ defined over $\mathbb{F}_q$ can also be described as $c^{(n)} = (c_1 \circ \mathrm{Fr}^n_{C,g}, c_2)$.

Definition 4.22 Let $f : X \to Y$ be a morphism of schemes and let Z be a closed subscheme of Y. Then $\mathscr{I}_{f^{-1}(Z)_{\text{red}}} = \sqrt{\mathscr{I}_{f^{-1}(Z)}}$ hence there exists $n \in \mathbb{N}$ such that $\mathscr{I}^n_{f^{-1}(Z)_{\text{red}}} \subset \mathscr{I}_{f^{-1}(Z)}$ (since X is noetherian). The smallest such n is called the *ramification of f at Z* and is denoted $\mathrm{ram}(f, Z)$.

Lemma 4.23 *Let $f : X \to Y$ be a quasi-finite morphism of schemes. Then, for every closed point $y \in Y$, $f^{-1}(y)$ is affine, corresponding to a finite $\mathbb{F}$-algebra A, and we have the inequality*

$$\mathrm{ram}(f, y) \le \dim_{\mathbb{F}} A.$$

Moreover, the set $\{\mathrm{ram}(f, y) \mid y \in Y \text{ closed}\}$ is bounded above.

Definition 4.24 For f quasi-finite as in the lemma we set

$$\mathrm{ram}(f) := \max\{\mathrm{ram}(f, y) \mid y \in Y \text{ closed}\}$$

and call this number the *ramification degree of f*.

Proof (4.23) Fix a closed point $y \in Y$. It is clear that $f^{-1}(y)$ is the spectrum of a finite $k(y) = \mathbb{F}$-algebra, say A. For the inequality set $d := \dim_{\mathbb{F}} A$. We then have to prove $\mathscr{I}^d_{f^{-1}(y)_{\text{red}}} \subset \mathscr{I}_{f^{-1}(y)}$. This is a local statement on both source and target and we may assume that $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$, y corresponding to a maximal ideal $\mathfrak{m} \subset C$. Then $A = B/\mathfrak{m}B$ is of $\mathbb{F} = C/\mathfrak{m}$-dimension d. We get a decreasing sequence of $\mathbb{F}$-vector spaces

$$B/\mathfrak{m}B \supset \sqrt{\mathfrak{m}B}/\mathfrak{m}B \supset (\sqrt{\mathfrak{m}B}/\mathfrak{m}B)^2 \supset \cdots,$$

which, by Nakayama's lemma, decreases strictly as long as the spaces are non-trivial. Hence the space $(\sqrt{\mathfrak{m}B}/\mathfrak{m}B)^d$ vanishes, which is equivalent to $\sqrt{\mathfrak{m}B}^d \subset \mathfrak{m}B$.

For the last statement, it suffices to prove that the $k(y)$-dimension of any fiber X_y over a closed point $y \in Y$ is uniformly bounded. Since this is obviously true if f is an open immersion we may assume f finite (by Zariski's main theorem). Also, since X is quasi-compact, we may assume $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$ are both affine. Now, B being a finite A-module, there is a surjective A-morphism $A^n \twoheadrightarrow B$, some $n \in \mathbb{N}$. Tensoring with $k(\mathfrak{p})$ over A ($\mathfrak{p} \subset A$ a prime ideal) shows that n bounds the dimensions of the fibers uniformly. □

Lemma 4.25 *Let $c : C \to X \times X$ be a correspondence defined over $\mathbb{F}_q$, let $Z \subset X$ be a closed subscheme and $n \in \mathbb{N}$ such that $q^n > \mathrm{ram}(c_2, Z)$ and Z is $c^{(n)}$-invariant. Then $c^{(n)}$ is contracting near Z.*

Proof Set $d = \mathrm{ram}(c_2, Z)$. We will prove more generally that for any coherent sheaf of $\mathscr{O}_C$-ideals $\mathscr{J}$ satisfying

(a) $c_1^{(n)*}(\mathscr{I}_Z) \cdot \mathscr{O}_C \subset \sqrt{\mathscr{J}}$ and

(b) $\sqrt{\mathscr{J}}^d \subset \mathscr{J}$, the following two inclusions hold:

1. $c_1^{(n)*}(\mathscr{I}_Z) \cdot \mathscr{O}_C \subset \mathscr{J}$;
2. $(c_1^{(n)*}(\mathscr{I}_Z) \cdot \mathscr{O}_C)^d \subset \mathscr{J}^{d+1}$.

(To obtain the lemma, apply this statement to $\mathscr{J} = c_2^*(\mathscr{I}_Z) \cdot \mathscr{O}_C$.)

To prove the more general claim we may assume $C = \operatorname{Spec}(B)$, $\mathscr{J}$ corresponding to an ideal $J \subset B$, $X = \operatorname{Spec}(A)$, and Z corresponding to an ideal $I \subset A$. Then we have

$$\begin{aligned} c_1^{(n)\sharp}(I) \cdot B &= \mathrm{Fr}_{C,g}^{n\sharp} \circ c_1^\sharp(I) \cdot B \\ &= \mathrm{Fr}_C^{n\sharp} \circ \mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I) \cdot B && \text{by (4.5)} \\ &= \mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I)^{q^n} \cdot B \\ &= (\mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I) \cdot B)^{q^n}. \end{aligned}$$

By (a), we thus have $(\mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I) \cdot B)^{q^n} \subset \sqrt{J}$ which implies $\mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I) \cdot B \subset \sqrt{J}$. We conclude:

$$\begin{aligned} c_1^{(n)\sharp}(I) \cdot B &= (\mathrm{Fr}_{C,a}^{-n\sharp} \circ c_1^\sharp(I) \cdot B)^{q^n} \\ &\subset \sqrt{J}^{q^n} \\ &\subset \sqrt{J}^{d+1} && \text{since } q^n \geq d+1 \\ &\subset J && \text{by (b).} \end{aligned}$$

The second inclusion is also easily obtained:

$$\begin{aligned} (c_1^{(n)\sharp}(I) \cdot B)^d &\subset \sqrt{J}^{d(d+1)} && \text{as above} \\ &\subset J^{d+1} && \text{by (b).} \end{aligned}$$

□

Corollary 4.26 *Let $c : C \to X \times X$ be a correspondence defined over $\mathbb{F}_q$.*

1. *Let $Z \subset X$ be a closed subscheme defined over $\mathbb{F}_q$ which is locally c-invariant. Then, for each $n \in \mathbb{N}$ with $q^n > \mathrm{ram}(c_2, Z)$, $c^{(n)}$ is contracting near Z in a neighborhood of fixed points.*
2. *If c_2 is quasi-finite then, for each $n \in \mathbb{N}$ with $q^n > \mathrm{ram}(c_2)$, $c^{(n)}$ is contracting near every closed point of X in a neighborhood of fixed points.*

Proof 1. First notice that Z is locally $c^{(n)}$-invariant. Indeed, let $x \in Z$. By assumption there exists an open neighborhood $U \subset X$ of x such that $Z \cap U$ is $c|^U$-invariant. Replacing c by $c|^U$ we may thus assume that Z is c-invariant. Shrinking X further we may assume $X = \operatorname{Spec}(A \otimes_{\mathbb{F}_q} \mathbb{F})$ for some $\mathbb{F}_q$-algebra A, Z corresponding to $I \otimes_{\mathbb{F}_q} \mathbb{F}$ for some ideal $I \subset A$. We will now prove that Z is $c^{(n)}$-invariant, i. e. $c_2^{-1}(Z)_{\mathrm{red}} \subset_s (c_1^{(n)})^{-1}(Z)$ (cf. 4.2). For this we may assume $C = \operatorname{Spec}(B \otimes_{\mathbb{F}_q} \mathbb{F})$ for some $\mathbb{F}_q$-algebra B, and $c = b \otimes_{\mathbb{F}_q} \mathbb{1}_{\mathbb{F}}$. Since Z is c-invariant we have $b_1^\sharp(I) \cdot B \otimes \mathbb{F} \subset \sqrt{b_2^\sharp(I) \cdot B \otimes \mathbb{F}}$ hence also $b_1^\sharp(I)^{q^n} \cdot B \otimes \mathbb{F} \subset \sqrt{b_2^\sharp(I) \cdot B \otimes \mathbb{F}}$. This proves local $c^{(n)}$-invariance of Z.

By 4.12.3, Z is also $c^{(n)}$-invariant in a neighborhood of fixed points hence, by 4.12.2, the open subset $W := W_{c^{(n)}}(Z) \subset C$ contains $\mathrm{Fix}(c^{(n)})$ and Z is $c^{(n)}|_W$-invariant. Clearly, $c^{(n)}|_W = (c|_W)^{(n)}$ and $\mathrm{ram}(c_2|_W, Z) \leq \mathrm{ram}(c_2, Z)$ hence 4.25 (applied to the correspondence $c|_W$) implies the claim.

2. As c_2 is quasi-finite every closed point of X is locally $c^{(n)}$-invariant by 4.5. Now, the claim may be deduced from 4.25 exactly as in part 1. □

§4 A generalization of Deligne's conjecture

Recall that in the construction of $\mathbf{R}\Gamma_c(u)$ for a correspondence c and a cohomological correspondence u we required that c_1 be proper (cf. 2.9). We now want to generalize this slightly in order to formulate the main result.

Definition 4.27 Let $c : C \to X_1 \times X_2$ be a correspondence, $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$, $i = 1, 2$, and let $j : U_1 \hookrightarrow X_1$ be an open subscheme such that $c_1|_{c_1^{-1}(U_1)} : c_1^{-1}(U_1) \to U_1$ is proper and $\mathscr{F}_1|_{X_1 \setminus U_1} = 0$. Given a cohomological correspondence $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$ we define a morphism

$$\mathbf{R}\Gamma_c{}^{U_1}(u) : \mathbf{R}\Gamma_c(X_1, \mathscr{F}_1) \longrightarrow \mathbf{R}\Gamma_c(X_2, \mathscr{F}_2)$$

as follows.

The map $\mathrm{adj} : \mathbf{R}\Gamma_c(U_1, \mathscr{F}_1|_{U_1}) \to \mathbf{R}\Gamma_c(X_1, \mathscr{F}_1)$ induced by the adjunction $j_! j^* \to \mathbb{1}_{X_1}$ is an isomorphism by assumption. Hence we may set $\mathbf{R}\Gamma_c{}^{U_1}(u)$ to be the composition

$$\mathbf{R}\Gamma_c(X_1, \mathscr{F}_1) \xrightarrow[\cong]{\mathrm{adj}^{-1}} \mathbf{R}\Gamma_c(U_1, \mathscr{F}_1|_{U_1}) \xrightarrow{\mathbf{R}\Gamma_c(u|^{U_1, X_2})} \mathbf{R}\Gamma_c(X_2, \mathscr{F}_2)$$

(cf. 2.13.2).

In the situation of the definition, suppose there exists $U_1 \subset V_1 \subset X$ open such that $c_1|_{c_1^{-1}(V_1)} : c_1^{-1}(V_1) \to V_1$ is still proper. We might wonder whether we then have $\mathbf{R}\Gamma_c{}^{U_1}(u) = \mathbf{R}\Gamma_c{}^{V_1}(u)$. It turns out that this is indeed the case.

Lemma 4.28 *Let $c : C \to X_1 \times X_2$ be a correspondence with c_1 proper, let $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$ and let $U_1 \subset X_1$ be an open subset such that $\mathscr{F}|_{X_1 \setminus U_1} = 0$. Then $\mathbf{R}\Gamma_c{}^{U_1}(u) = \mathbf{R}\Gamma_c(u)$ for every $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$.*

Proof Set $[j] = [j^{U_1, X_2}]$. It will suffice to prove that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{R}\Gamma_c(U_1, j_1^* \mathscr{F}_1) & & \\
\downarrow \cong & \searrow^{\mathbf{R}\Gamma_c([j]^* u)} & \\
\mathbf{R}\Gamma_c(X_1, j_{1!} j_1^* \mathscr{F}_1) & \xrightarrow{\mathbf{R}\Gamma_c([j]_! [j]^* u)} & \mathbf{R}\Gamma_c(X_2, \mathscr{F}_2) \\
\mathrm{adj} \downarrow \cong & \nearrow_{\mathbf{R}\Gamma_c(u)} & \\
\mathbf{R}\Gamma_c(X_1, \mathscr{F}_1) & &
\end{array}$$

By 2.10, the upper half commutes. For the lower half we have to compare $[j]_! \circ [j]^*(u)$ and u. The former equals the composition of the dotted arrows in the following diagram ($d = [j]^* c$):

$$\begin{array}{ccccc}
c_{2!} c_1^* j_{1!} j_1^* \mathscr{F}_1 & \xrightarrow{(2.3)} & c_{2!} j_!^{\natural} d_1^* j_1^* \mathscr{F}_1 & \xrightarrow[\cong]{(1.13)} & c_{2!} c_1^* j_{1!} j_1^* \mathscr{F}_1 & \\
 & & \cong \downarrow & & \downarrow \mathrm{adj} & \\
 & & d_{2!} j^{\natural *} c_1^* \mathscr{F}_1 & \xrightarrow[(1.11)]{} & c_{2!} c_1^* \mathscr{F}_1 & \xrightarrow[u]{} \mathscr{F}_2
\end{array}$$

The first thing to note is that the morphism (1.11) appearing in the diagram is induced by the adjunction morphism adj : $j^{\natural}_! j^{\natural *} \to \mathbb{1}$. Next, this as well as the other morphism denoted adj are equal to the trace morphism defined in [2, XVII, 6.2.3]. Hence, by [2, XVII, 6.2.3, (Var 2)], the square above commutes.

Furthermore, since $j^{\natural}$ and j_1 are open immersions, the morphism (2.3) is easily seen to be the inverse of (1.13). We conclude that $[j]_! \circ [j]^*(u) = u \circ \mathrm{adj}$. Applying $\mathbf{R}\Gamma_c$ on both sides, the claimed commutativity follows immediately. □

Returning to the situation before the lemma, we have

$$\begin{aligned}
\mathbf{R}\Gamma_c{}^{U_1}(u) &= \mathbf{R}\Gamma_c(u|^{U_1,X_2}) \circ \mathrm{adj}^{-1}_{U_1 \hookrightarrow X_1} \\
&= \mathbf{R}\Gamma_c((u|^{V_1,X_2})|^{U_1,X_2}) \circ \mathrm{adj}^{-1}_{U_1 \hookrightarrow V_1} \circ \mathrm{adj}^{-1}_{V_1 \hookrightarrow X_1} && \text{by 2.12} \\
&= \mathbf{R}\Gamma_c{}^{U_1}(u|^{V_1,X_2}) \circ \mathrm{adj}^{-1}_{V_1 \hookrightarrow X_1} \\
&= \mathbf{R}\Gamma_c(u|^{V_1,X_2}) \circ \mathrm{adj}^{-1}_{V_1 \hookrightarrow X_1} && \text{by 4.28} \\
&= \mathbf{R}\Gamma_c{}^{V_1}(u).
\end{aligned}$$

This equality (together with the fact that if $(c|_{c_1^{-1}(U_1)})_1$ and $(c|_{c_1^{-1}(U_1')})_1$ are both proper then so is $(c|_{c_1^{-1}(U_1 \cup U_1')})_1$) justifies the following notational convention.

Notation 4.29 Let $c : C \to X_1 \times X_2$ be a correspondence, $\mathscr{F}_i \in \mathfrak{D}^b_{ctf}(X_i)$, and $u \in \mathrm{Hom}_c(\mathscr{F}_1, \mathscr{F}_2)$. We set $\mathbf{R}\Gamma_c(u)$ to be $\mathbf{R}\Gamma_c{}^{U_1}(u)$ for any open $U_1 \subset X_1$ such that $c_1|_{c_1^{-1}(U_1)} : c_1^{-1}(U_1) \to U_1$ is proper and $\mathscr{F}_1|_{X_1 \setminus U_1} = 0$, provided such a U_1 exists.

We are now ready to formulate and prove our main theorem.

Theorem 4.30 *Let $c : C \to X \times X$ be a correspondence defined over $\mathbb{F}_q$.*

1. *Assume that c_2 is quasi-finite. Then for every $n \in \mathbb{N}$ with $q^n > \mathrm{ram}(c_2)$, the space $\mathrm{Fix}(c^{(n)})$ is finite and discrete.*
2. *Let $U \subset X$ be an open subset defined over $\mathbb{F}_q$ such that $c_1|_{c_1^{-1}(U)}$ is proper, $c_2|_{c^{-1}(U \times U)}$ is quasi-finite, and $Z := X \setminus U$ is locally c-invariant. Endow Z with any closed subscheme structure. Then there exists a $d \in \mathbb{N}$ with the following property:*

   For every $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$ with $\mathscr{F}|_{X \setminus U} = 0$, every $n \in \mathbb{N}$ with $q^n > d$ and every cohomological correspondence $u \in \mathrm{Hom}_{c^{(n)}}(\mathscr{F}, \mathscr{F})$, we have an equality

$$\mathrm{Tr}(\mathbf{R}\Gamma_c(u)) = \sum_{y \in \mathrm{Fix}(c^{(n)}) \cap c^{-1}(U \times U)} \mathrm{Tr}(u_y). \tag{4.6}$$

3. *In the notation of 2, assume that X and C are both proper. Then*

$$d := \max\{\mathrm{ram}(c_2|_{c^{-1}(U \times U)}), \mathrm{ram}(c_2, Z)\}$$

   satisfies the conclusion of 2.

4. *In the notation of 2, let $\overline{c} : \overline{C} \to \overline{X} \times \overline{X}$ be a compactification of $c|_{c_1^{-1}(U)}$ such that $\overline{Z} := \overline{X} \setminus U$ is locally $\overline{c}$-invariant (cf. 4.13). Endow $\overline{Z}$ with any closed subscheme structure. Then*

$$d := \max\{\mathrm{ram}(\overline{c}_2|_{\overline{c}^{-1}(U \times U)}), \mathrm{ram}(\overline{c}_2, \overline{Z})\}$$

   satisfies the conclusion of 2.

Proof 1. Let β be a connected component of $\mathrm{Fix}(c^{(n)})$. We shall prove that β is a point. Our general assumptions on schemes ensure the existence of a closed point $x \in (c^{(n)})'(\beta)$. By 4.26.2, $c^{(n)}$ is contracting near x in a neighborhood of fixed points. By 4.18.1, β is an open connected subscheme of $(c^{(n)})'^{-1}(x) \subset c_2^{-1}(x)$ which is a finite scheme hence β is a point.

3. Thus assume X and C (hence also c) proper. Let $\mathscr{F} \in \mathfrak{D}^b_{ctf}(X)$, $n \in \mathbb{N}$ such that $q^n > d$ and $u \in \mathrm{Hom}_{c^{(n)}}(\mathscr{F}, \mathscr{F})$. We will deduce equality (4.6) from the Lefschetz-Verdier trace formula (3.20). The latter yields an equality

$$\mathrm{Tr}(\mathbf{R}\Gamma_c(u)) = \sum_{\beta \in \pi_0(\mathrm{Fix}(c^{(n)}))} \mathrm{lt}_\beta(u).$$

By 4.28, the left hand side equals the left hand side of (4.6). For the right hand side let $\beta \in \pi_0(\mathrm{Fix}(c^{(n)}))$. There are two cases to consider.

Case 1 First assume $c(\beta) \not\subset U \times U$, let $x \in \beta$ such that $c(x) \notin U \times U$. By 4.26.1, $c^{(n)}$ is contracting near Z in a neighborhood of fixed points. Also, $(c^{(n)})'(x) = c'(x) \in Z$ hence $(c^{(n)})'(\beta) \cap Z \neq \varnothing$ and $c^{(n)}$ satisfies the hypotheses of 4.18. Hence β is a connected component of $\mathrm{Fix}(c^{(n)}||^Z)$ and $\mathrm{lt}_\beta(u) = \mathrm{lt}_\beta(u||^Z)$. But $\mathscr{F}|_Z = 0$ hence $u||^Z = 0$ which implies $\mathrm{lt}_\beta(u) = 0$.

Case 2 On the other hand, assume $c(\beta) \subset U \times U$. Denote by d the correspondence $c|_{c^{-1}(U\times U)}$. Then $d_2 = c_2|_{c^{-1}(U\times U)}$ is quasi-finite and $q^n > d \geq \mathrm{ram}(d_2)$ hence, by part 1, the set $\mathrm{Fix}(d^{(n)})$ is finite. Since $d^{(n)} = c^{(n)}|_{c^{-1}(U\times U)}$, we have $\beta \subset \mathrm{Fix}(c^{(n)}) \cap c^{-1}(U \times U) = \mathrm{Fix}(d^{(n)})$ from which it follows that $\beta = y$ is a closed point and that the sum on the right hand side of (4.6) is finite. We are done if we can prove $\mathrm{lt}_y(u) = \mathrm{Tr}(u_y)$.

Set $x = c_2(y) = d_2(y) \in X$. By 4.26.2, $d^{(n)}$ is contracting near x in a neighborhood of fixed points (that x is closed may be proved as in 4.5), hence 4.18.2 tells us that $\mathrm{lt}_y(u|_{c^{-1}(U\times U)}) = \mathrm{lt}_y(u|_{c^{-1}(U\times U)}||^x)$. We conclude:

$$\begin{aligned}
\mathrm{lt}_y(u) &= \mathrm{lt}_y(u|_{c^{-1}(U\times U)}) && \text{by 3.22}\\
&= \mathrm{lt}_y(u|_{c^{-1}(U\times U)}||^x)\\
&= \mathrm{Tr}(u_y) && \text{by 4.11.}
\end{aligned}$$

2. If we replace c and u by its restrictions $c|_{c_1^{-1}(U)}$ and $u|_{c_1^{-1}(U)}$, respectively, then the hypotheses of 2 are still satisfied. Moreover, if we can prove the existence of such a $d \in \mathbb{N}$ with respect to $c|_{c_1^{-1}(U)}$ and $u|_{c_1^{-1}(U)}$ then the same d also works for c and u since (4.6) remains the same. Indeed, for the left hand side we have:

$$\begin{aligned}
\mathrm{Tr}(\mathbf{R}\Gamma_c{}^U(u)) &= \mathrm{Tr}(\mathbf{R}\Gamma_c(u|^{U,X}) \circ \mathrm{adj}^{-1})\\
&= \mathrm{Tr}(\mathbf{R}\Gamma_c((u|_{c_1^{-1}(U)})|^{U,X}) \circ \mathrm{adj}^{-1}) && \text{by 2.12}\\
&= \mathrm{Tr}(\mathbf{R}\Gamma_c{}^U(u|_{c_1^{-1}(U)})).
\end{aligned}$$

For each y on the right hand side the equality

$$\begin{aligned}
\mathrm{Tr}(u_y) &= \mathrm{Tr}((u|_{c^{-1}(U\times U)})_y) && \text{by definition, cf. 4.9.2}\\
&= \mathrm{Tr}((u|_{c_1^{-1}(U)}|_{c^{-1}(U\times U)})_y) && \text{by 2.12}\\
&= \mathrm{Tr}((u|_{c_1^{-1}(U)})_y) && \text{by definition, cf. 4.9.2,}
\end{aligned}$$

shows that the sum remains the same as well. Hence we may assume without loss of generality that $c_1^{-1}(U) = C$. We will deduce part 2 from 3.

4.13 ensures the existence of a compactification $\overline{c} : \overline{C} \to \overline{X} \times \overline{X}$ of c such that $\overline{X} \backslash U$ is locally $\overline{c}$-invariant. Denote by $[j]$ the inclusion of $c|^{U,X}$ into $\overline{c}$. Since $c|_1^{U,X}$ is proper, $[j]$ satisfies (F3) of 2.§3 hence we may set $\overline{u} = [j]_!(u|^{U,X}) \in \mathrm{Hom}_{\overline{c}}(j_{2!}\mathscr{F}, j_{2!}\mathscr{F})$, where we used the identification $\overline{\mathscr{F}} := j_{1!}\mathscr{F}|_U \cong j_{2!}\mathscr{F}$ since $\mathscr{F}_{X\backslash U} = 0$. By 4.13.1, $\overline{c}_1^{-1}(U) = c_1^{-1}(U) \subset C$ hence also $\overline{c}^{-1}(U \times U) = c^{-1}(U \times U)$ which implies that the pair $(\overline{c}, U)$ (replacing (c, U)) satisfies the hypotheses of part 3. It thus suffices to show that neither the left nor the right hand side of the equality (4.6) change under the replacement $(c, U, \mathscr{F}, u) \mapsto (\overline{c}, U, \overline{\mathscr{F}}, \overline{u})$.

For the left hand side we have

$$\begin{aligned} \mathrm{Tr}(\mathbf{R}\Gamma_c{}^U(u)) &= \mathrm{Tr}(\mathbf{R}\Gamma_c(u|^{U,X}) \circ \mathrm{adj}^{-1}) \\ &= \mathrm{Tr}(\mathbf{R}\Gamma_c(\overline{u}) \circ \mathrm{adj}^{-1}) && \text{by 2.10} \\ &= \mathrm{Tr}(\mathbf{R}\Gamma_c(\overline{u})). \end{aligned}$$

On the right hand side of (4.6), the set of y over which the sum runs clearly does not change. And for each such y we have the following equality ($x = c_2(y)$):

$$\begin{aligned} \mathrm{Tr}(u_y) &= \mathrm{Tr}(u|_{W_c(x)}|^x|_y) \\ &= \mathrm{Tr}(u|_{W_c(x)}|^{U,X}|^{x,x}|_y) && \text{by 2.15 since } c_1^{-1}(U) = C \\ &= \mathrm{Tr}(u|^{U,X}|_{W_c(x)}|^{x,x}|_y) && \text{by 2.12} \\ &= \mathrm{Tr}(\overline{u}|_C|^{U,X}|_{W_c(x)}|^{x,x}|_y) && \text{by 2.16 and 2.12} \\ &= \mathrm{Tr}(\overline{u}|_{W_c(x)}|^{U,X}|^{x,x}|_y) && \text{by 2.12} \\ &= \mathrm{Tr}(\overline{u}|_{W_c(x)}|^x|_y) && \text{by 2.15 since } c_1^{-1}(U) = C \\ &= \mathrm{Tr}(\overline{u}|_{W_{\overline{c}}(x)}|^x|_{W_c(x) \cap \overline{c}_2^{-1}(x)_{\mathrm{red}}}|_y) && \text{by 2.14} \\ &= \mathrm{Tr}(\overline{u}|_{W_{\overline{c}}(x)}|^x|_y) && \text{by 2.12} \\ &= \mathrm{Tr}(\overline{u}_y). \end{aligned}$$

4. This is clear from the proof of part 2. □

Notation

Term	Page of definition
adj	5
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$\mathrm{bc}^*_{\cdot}, \mathrm{bc}^!_{\cdot}, \mathrm{bc}_{*\cdot}, \mathrm{bc}_{!\cdot}$	48, 49
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$\boxtimes$	8
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$\tilde{\cdot}.$	34
$\cdot^{\circledast}$	4
c_k	13
$C.$	35
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cor	58
cor′	48
$\cdot^{(n)}$	91
$[\cdot]^*$	18
$[\cdot]_!$	15
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$d.$	49
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$\mathfrak{D}$	4
$\mathfrak{D}^b_{ctf}$	4
$\mathfrak{D}^b_{ctf}\mathfrak{f}$	77
$\mathfrak{D}\mathfrak{f}$	11
$\mathbb{D}.$	5
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ε'	7
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$L.$	70
l	4
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$\cdot'$	34
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Q	50
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$R.$	70
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$\mathbf{R}\Gamma_{\mathrm{c}}$	5, 17, 95
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$\cdot_{\mathrm{red}}$	4
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$\lvert\rvert^{\cdot}$	84
$\cdot\lvert_{\cdot}, \cdot\rvert^{\cdot,\cdot}$	5, 13, 14, 18
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s	9
$=_s, \subset_s$	83
Sch/·	4
sCor	13
SCor	24
$\cdot^{\sharp}$	4
$\mathrm{sp}_{\cdot}$	22, 24, 25
$t.$	48
$=_t, \subset_t$	83
$\tau^{[\cdot,\cdot]}$	11
$t_{\cdot *}, t_{\cdot_*}, t_{\cdot !}$	6
Tr	47
tr.	46
$W.$	84

References

[1] Michael Artin, Alexandre Grothendieck, and Jean-Louis Verdier. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 4.2: Théorie des Topos et Cohomologie Étale des Schémas*, volume 270 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 1973.

[2] Michael Artin, Alexandre Grothendieck, and Jean-Louis Verdier. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 4.3: Théorie des Topos et Cohomologie Étale des Schémas*, volume 305 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 1973.

[3] Pierre Berthelot, Alexandre Grothendieck, and Luc Illusie. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 6: Théorie des intersections et théorème de Riemann-Roch*, volume 225 of *Lecture Notes in Mathematics*. Springer, Berlin, New York, 1971.

[4] Nicolas Bourbaki. *Algèbre commutative. Chap. 6–7: Entiers – Valuations*. Éléments de mathématique. Hermann, Paris, 1964.

[5] Brian Conrad. Deligne's Notes on Nagata Compactifications. *Journal of the Ramanujan Mathematical Society*, 22(3):205–57, 2007.

[6] Pierre Deligne. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 4½: Cohomologie étale*, volume 569 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 1977.

[7] Pierre Deligne. La conjecture de Weil II. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980.

[8] Pierre Deligne and Nicholas Katz. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 7.2: Groupes de monodromie en géométrie algébrique*, volume 340 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 1973.

[9] Kazuhiro Fujiwara. Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture. *Inventiones mathematicae*, 127:489–533, 1997.

[10] William Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin Heidelberg, 1984.

[11] Alexandre Grothendieck. Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes. *Publications Mathématiques de l'IHÉS*, 8, 1961. rédigés avec la collaboration de Jean Dieudonné.

[12] Alexandre Grothendieck. Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, seconde partie. *Publications Mathématiques de l'IHÉS*, 24, 1965. rédigés avec la collaboration de Jean Dieudonné.

[13] Alexandre Grothendieck. Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, quatrième partie. *Publications Mathématiques de l'IHÉS*, 32, 1967. rédigés avec la collaboration de Jean Dieudonné.

[14] Alexandre Grothendieck, I. Bucur, C. Houzel, Luc Illusie, J.-P. Jouanolou, and J.-P. Sierre. *Séminaire de Géométrie Algébrique du Bois-Marie SGA 5: Cohomologie l-adique et Fonctions L*, volume 589 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 1977.

[15] Luc Illusie. *Complexe cotangent et déformations. I*, volume 239 of *Lecture Notes in Mathematics*. Springer, 1971.

[16] Luc Illusie. Autour du théorème de monodromie locale. In *Périodes p-adiques*, number 223 in Astérisque, pages 9–57. Société mathématique de France, 1994.

[17] Masayoshi Nagata. On the theory of henselian rings. *Nagoya Mathematical Journal*, 5:45–57, 1953.

[18] Richard Pink. On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne. *The Annals of Mathematics, The Second Series*, 135(3):483–525, May 1992.

[19] Yakov Varshavsky. Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara. *Geometric and Functional Analysis*, 17:271–319, 2007.

[20] Jean-Louis Verdier. Spécialisation de faisceaux et monodromie modérée. In *Analyse et topologie sur les éspaces singuliers*, number 101–102 in Astérisque, pages 332–64. Société mathématique de France, 1983.